LECTURE 1.

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In the first lecture, we mainly introduced new definitions and mentioned several examples. In this note, I will just highlight what we did.

Warning: Reading these notes is not enough by any means. You have to also read your book.

(1) \((G, \cdot)\) is a semigroup if
   (a) \(G\) is closed under multiplication, i.e.
       \[ \forall a, b \in G, a \cdot b \in G. \]
   (b) \(\cdot\) is associative, i.e.
       \[ \forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c). \]

(2) \((G, \cdot)\) is a monoid if
   (a) \((G, \cdot)\) is a semigroup.
   (b) \(G\) has an identity, i.e.
       \[ \exists e \in G, \forall g \in G, eg = ge = g. \]

(3) \((R, +, \cdot)\) is a ring if
   (a) \((R, +)\) is a commutative group.
   (b) \((R, \cdot)\) is a semigroup.
   (c) (Distribution property) \[ \forall a, b, c \in R, a(b + c) = ab + ac \& (b + c)a = ba + ca. \]

(4) A ring \(R\) is called unital if \((R, \cdot)\) is a monoid.
(5) A ring \(R\) is called commutative if \[ \forall a, b \in R, ab = ba. \]
(6) Let \(R\) be a unital ring. \(a\) is called a left-inverse of \(b\) if \(ab = 1\). Similarly one can defined a right-inverse.
(7) Let \(R\) be a unital ring. \(a \in R\) is called invertible or a unit if it has both a left-inverse and a right-inverse. The set of all the units in \(R\) is denoted by \(U(R)\).

Here are some of the examples that we discussed:

(1) \(\mathbb{Z}, \times\) is a monoid but it is not a group.
(2) \(\mathbb{Z} \setminus \{1\}, \times\) is not a semigroup.
(3) \(2\mathbb{Z}, \times\) is a semigroup but it is not a monoid.
(4) \(\mathbb{Z}, +, \times\) is a ring.
(5) \(\mathbb{N}, +, \times\) is not a ring.
(6) The set \(M_n(\mathbb{R})\) of \(n\)-by-\(n\) real matrices form a non-commutative ring.
(7) \(U(M_n(\mathbb{R})) = GL_n(\mathbb{R})\).
(8) \((GL_n(\mathbb{R}), +, \cdot)\) is not a ring!
(9) If \(x \in M_n(\mathbb{R})\) has a left-inverse, then it is invertible.

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