Lemma 1. Let $I$ be an ideal of $R$. Consider the abelian additive group $R/I$. Then the following is a well-defined operation

$$(a + I) \cdot (b + I) := (ab) + I.$$ 

Moreover $(R/I, +, \cdot)$ is a ring.

Proof. Well-defined: we have to show that if $a + I = a' + I$ and $b + I = b' + I$, then $(ab) + I = (a'b') + I$. It is equivalent to say that if $a - a' \in I$ and $b - b' \in I$, then $ab - a'b' \in I$:

$$ab - a'b' = (ab - ab') + (ab' - a'b') = a(b - b') + (a - a')b' \in RI + IR \subseteq I.$$ 

It is straightforward to check that it is a ring. □

Corollary 2. Let $f : R \to R/I$, $f(a) := a + I$. Then $f$ is a ring homomorphism and $\ker(f) = I$.

Proof. It is a direct corollary of Lemma 1 that $f$ is a ring homomorphism. We also have

$$\ker(f) := \{a \in R | f(a) = 0\} = \{a \in R | a + I = 0 + I\} = I.$$ 

Corollary 3. There is a correspondence between ideals and kernels of ring homomorphisms.

Definition 4. A ring homomorphism $f : R \to S$ is called an isomorphism if it is a bijection.

Lemma 5. If $f : R \to S$ is a ring isomorphism, then $f^{-1} : S \to R$ is also an isomorphism.

Proof. Since $f^{-1}$ is clearly a bijection, it is enough to prove that it is a ring homomorphism:

$$f(f^{-1}(x) + f^{-1}(y)) = f(f^{-1}(x)) + f(f^{-1}(y)) = x + y$$

and

$$f(f^{-1}(x)f^{-1}(y)) = f(f^{-1}(x))f(f^{-1}(y)) = xy.$$ 

Hence

$$f^{-1}(x) + f^{-1}(y) = f^{-1}(x + y) \text{ and } f^{-1}(x)f^{-1}(y) = f^{-1}(xy).$$ □

Lemma 6. $f \in \text{hom}(R, S)$ is an isomorphism if and only if $\text{Im}(f) = S$ and $\ker(f) = \{0\}$.

Proof. It is enough to show that a homomorphism is injective if and only if $\ker(f) = \{0\}$.

If $x \in \ker(f)$, then $f(x) = f(0)$. So if $f$ is injective, then $x = 0$.

If $f(x) = f(y)$, then $f(x - y) = 0$, which means $x - y \in \ker(f)$. So if $\ker(f) = \{0\}$, then $x - y = 0$, i.e. $x = y$. Hence $f$ is injective. □