More on group actions and quotients

Tuesday, May 16, 2017 11:42 AM

Theorem. Let G be a connected affine algebraic group.

Suppose X is a G-homogeneous variety, i.e. $G \curvearrowright X$ algebraically and transitively. Then

① X is irreducible and smooth.

② Suppose $G \curvearrowright X, Y$ and $X \xrightarrow{\phi} Y$ is G-equivariant. Then
Y: homogen.

ϕ is separable $\iff d\phi_x$ is surjective for some $x \in X$

$\iff d\phi_x$ is surjective for any $x \in X$

③ $G_1 \xrightarrow{\phi} G_2$ surjective affine algebraic group homomorphism.

ϕ is separable $\iff d\phi_e$ is surjective.

Pf. ① $\left.\begin{array}{l} G \longrightarrow X \\ g \longmapsto g.x_0 \text{ is onto} \\ G \text{ : irreducible} \end{array}\right\} \Rightarrow X$ is irreducible.

$\exists\ x \in X$ which is simple $\Rightarrow g x_0$ is simple, $\forall g \in G$

as $\begin{array}{l} X \longrightarrow X \\ x \longmapsto g.x \end{array}$ is an isomorphism.

$\Rightarrow X$ is smooth.

② Since Y is homog., ϕ is dominant. So

ϕ is separable $\Leftrightarrow \exists\, x_0 \in X$ s.t. ① x_0 and $\phi(x_0)$ are simple
② $d\phi_{x_0}$ is surjective

Since X and Y are smooth, we get

ϕ is separable $\Leftrightarrow \exists\, x_0 \in X$, $d\phi_{x_0}$ is surjective.

By homog., we get that for any $x \in X$, $d\phi_x$ is surjective. ∎

③ is a corollary of part ②. ∎