Ricci flow on Wallach flag varieties

Nolan R. Wallach

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The curvature

- If $x_1 = x_2$ then the sectional curvature is strictly positive if $0 < \frac{x_3}{x_1} < 1$ or $1 < \frac{x_3}{x_1} < \frac{4}{3}$ and there is some strictly negative curvature if $\frac{x_3}{x_1} > \frac{4}{3}$.
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- The symmetric group acting by permuting factors preserves positive curvature. We consider the case when $x_3 < x_1 < x_2$. Since scaling by a constant preserves the sign of curvature we consider $x_1 = 1$, $x_2 = 1 + r$ and $x_3 = s$ with $r > 0$ and $0 < s < 1$. 

With the notation above a necessary and sufficient condition that the sectional curvature be positive is $r < s^2 + 2p_1s + s^2 < s^2$ (equivalent to Valiev's result).

We note that if $0 < s < 1$ then $s^2 + 2p_1s + s^2 < s^2$. 

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- With the notation above a necessary and sufficient condition that the sectional curvature be positive is $r < \frac{s^2 - 2 + 2\sqrt{1 - s + s^2}}{3}$ (equivalent to Valiev’s result).

- We note that if $0 < s < 1$ then

$$\frac{s^2}{4} < \frac{s - 2 + 2\sqrt{1 - s + s^2}}{3} < \frac{s^2}{3}.$$
Fundamental domain for $S_3$ acting on the homogeneous metrics of positive curvature consists of the points in the first quadrant below the graph the sets \( \{(s, 0)| 0 < s < 1\} \) and \( \{(1, r)|0 < r < \frac{1}{3}\} \).
\[ \text{Ric}(g) = x_1 r_1 \langle \ldots, \ldots \rangle_1 + x_2 r_2 \langle \ldots, \ldots \rangle_2 + x_3 r_3 \langle \ldots, \ldots \rangle_3. \]
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\[ r_i = \frac{dx_i^2 - dx_j^2 - dx_k^2 + (10d - 8)x_jx_k}{2x_1x_2x_3} \]

where \( \{i, j, k\} = \{1, 2, 3\} \).
Hamilton’s Ricci flow is given in these parameters as

$$\frac{dx_i}{dt} = -2r_i x_i.$$
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The goal is to say what happens to positive sectional curvature or Ricci curvature under the above non-linear ODE.
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- A direct calculation shows that for $d = 2, 4, 8$

\[
\frac{d}{dt} \frac{x_3(t)}{x_1(t)} = -2 \frac{x_3(t)}{x_1(t)} (r_3 - r_1) = \frac{-2d(1 - \frac{x_3}{x_1})(4 \frac{(d-1)}{d} - \frac{x_3}{x_1})}{x_1^2}.
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- Hence if $0 < \frac{x_3(t)}{x_1(t)} < 1$ then $\frac{d}{dt} \frac{x_3(t)}{x_1(t)} < 0$, if $1 < \frac{x_3(t)}{x_1(t)} < 4 \frac{d-1}{d}$ then $\frac{d}{dt} \frac{x_3(t)}{x_1(t)} > 0$ and if $\frac{x_3(t)}{x_1(t)} > 4 \frac{d-1}{d}$ then $\frac{d}{dt} \frac{x_3(t)}{x_1(t)} < 0$. That is the line through $1, 1, 1$ is repelling fixed point and that through $1, 1, 4 \frac{d-1}{d}$ is an attractor.
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- The lines through $1, 1, 1$ and $1, 1, 4\frac{d-1}{d}$ give the full set of Einstein metrics among the metrics with $x_1 = x_2$. 
This implies that if \( 1 < \frac{x_1(0)}{x_3(0)} < 4 \frac{d-1}{d} \) then we have
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\lim_{t \to +\infty} \frac{x_1(t)}{x_3(t)} = 4 \frac{d-1}{d}
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**Theorem**

*For all three examples the Ricci flow deforms certain metrics of strictly positive sectional curvature to metrics with some strictly negative sectional curvature.*
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**Theorem**

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We also note that since the Ricci tensor is positive definite for \(1, 1, s\) and \(0 < s \leq 4 \frac{d-1}{d}\) this implies that the flow cannot change the signature of the Ricci tensor if it starts with strictly positive curvature and \(x_1 = x_2\).
Ricci curvature

We assume \( x_2 > x_1 > x_3 > 0 \) and scale to \( x_1 = 1, x_2 = 1 + r, x_3 = s \) with \( r > 0 \) and \( 0 < s < 1 \).

\[
\begin{align*}
 r_1 x_1 &= \frac{-2rd - dr^2 + (10d - 8)s + (10d - 8)rs - ds^2}{2(1 + r)s}, \\
 r_2 x_2 &= \frac{dr + dr^2 + (10d - 8)s - ds^2}{2s}, \\
 r_3 x_3 &= \frac{(8d - 8) + (8d - 8)r - dr^2 + ds^2}{2(1 + r)}.
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- If $0 < r < 8$ only the first can change sign: positive definite Ricci curvature if and only if

$$r < \sqrt{1 + 8s^2} - (1 - 3s), d = 2$$

$$r < \sqrt{1 + 15s^2} - (1 - 4s), d = 4$$

$$r < \sqrt{1 + \frac{77}{4}s^2} - \left(1 - \frac{9}{2}s\right), d = 8.$$  

since all of these expressions are $< 8$ if $0 < s < 1$. 

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To change the signature we start with a point with \( r_1 = 0 \) and hope that 
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\[
0 < s < 1 - \sqrt{\frac{5}{8}} = 0.20943058...
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\[
0 < s < \frac{30 + 5\sqrt{21} - 3\sqrt{5(21 + 4\sqrt{21})}}{30} = 0.361437...
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0 < s < \frac{693 + 11\sqrt{2737} - 7\sqrt{22(511 + 9\sqrt{2737})}}{616} = 0.389089...
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Theorem
For all the examples the Ricci flow of a metric with positive definite Ricci tensor can flow to one with signature \( (d, 2d) \).
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For all the examples the Ricci flow of a metric with positive definite Ricci tensor can flow to one with signature \((d, 2d)\).
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There exist homogeneous metrics of strictly positive sectional curvature on the 12 and 24 dimensional examples that deform under the Ricci flow to metrics with some negative Ricci curvature.
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**Theorem**

If $g_0$ is a homogeneous Riemannian structure on the 6 dimensional example with strictly positive sectional curvature then under the Ricci flow it retains strictly positive Ricci curvature.
We continue with the assumption $x_2 > x_1 > x_3 > 0$ so $\frac{x_2}{x_1} = 1 + r$ and $\frac{x_3}{x_1} = s$ with $r > 0$ and $0 < s < 1$. 
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\[ r' = \frac{-2(x_2'x_1 - x_1'x_2)}{x_1^2} = 2(1 + r)(r_1 - r_2) = g(d, r, s) \]
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s' = \frac{-2(x'_3 x_1 - x'_1 x_3)}{x_1^2} = h(d, r, s).
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\[
g(d, r, s) = \begin{cases} 
-4 \frac{r}{s}(2 + r - 3s), & d = 2 \\
-8 \frac{r}{s}(2 + r - 4s), & d = 4 \\
-8 \frac{r}{s}(4 + 2r - 9s), & d = 8 
\end{cases}
\]
The function $h(d, r, s)$ can be expressed as:

$$h(d, r, s) = \begin{cases} 
4 \left( \frac{1-s}{1+r} \right)(-2 - 3r + s), & d = 2 \\
8 \left( \frac{1-s}{1+r} \right)(-3 - 4r + s), & d = 4 \\
8 \left( \frac{1-s}{1+r} \right)(-7 - 9r + 2s), & d = 8
\end{cases}.$$
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If \(0 < s < 1\) and \(r > 0\) then \(h(d, r, s) < 0\). We can thus think of \(r\) as a function of \(s\) in this range and have

\[ r'(s) = \frac{r'(t)}{s'(t)} = \frac{r}{s} f(d, r, s) \]
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\[ f(d, r, s) = \frac{g(d, r, s)}{h(d, r, s)} = \frac{1 + r}{1 - s} \begin{cases} 
\frac{2 + r - 3s}{2 + 3r - 5s}, & d = 2 \\
\frac{2 + r - 4s}{3 + 4r - 7s}, & d = 4 \\
\frac{4 + 2r - 9s}{7 + 9r - 2s}, & d = 8 
\end{cases} \]
Lemma

Suppose that we have a solution to the Ricci flow with initial condition $s_0 > 0$, $r(s_0) > 0$ and $r(s)$ is defined for $0 < s_1 \leq s \leq s_o$.

1. If $f(d, r(s), s) \geq C > 0$ in this range then we have

$$r(s) \leq s^C \frac{r(s_0)}{s_0^C}, s_1 \leq s \leq s_o.$$

2. If $0 < f(d, r(s), s) \leq C$ in this range we have

$$r(s) \geq s^C \frac{r(s_0)}{s_0^C}, s_1 \leq s \leq s_o.$$
**Lemma**

If \( d = 2 \) then \( r_2, r_3 > 0 \) if \( 0 < s < 1 \) and \( 0 < r < 2(1 + \sqrt{2}) \).

**Lemma**

If \( d = 2, 0 < s < 1 \) and \( r(s) > s \) then \( r'(s) > 0 \). Suppose that \( 0 < s_0 < 1, s_0 < r(s_0) \leq 2s_0 \) and \( 0 < s_1 < s_0 \) is such that \( r(s) \) is defined and \( r(s) > s \) for \( s_1 \leq s \leq s_0 \). Then \( r(s) < 2s \).

The point here is that the smallest value of \( C \) in the calculus lemma is 1.
We have seen that the condition for some negative Ricci curvature is
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The Lemma above implies that under this condition \( r(s) \) can never pass \( 2s \).
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The Lemma above implies that under this condition \( r(s) \) can never pass \( 2s \).

This completes the argument for the case \( d = 2 \).
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One finds that in these cases one can take $C = \frac{5}{6}$ so

$$r(s) \geq s^{\frac{5}{6}} \text{Const.}$$

for $s$ sufficiently small and since $s \to 0$ along the Ricci flow hence along the flow $\frac{r}{s}$ becomes arbitrarily large.