MEASURE RIGIDITY FOR THE ACTION OF SEMISIMPLE GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. We classify measures which are invariant under the action of semisimple groups in arbitrary characteristic.

1. INTRODUCTION

Let $T$ be a finite set and let $k_\nu$ be a local field for all $\nu \in T$. Put $k_T = \prod_{\nu \in T} k_\nu$. We fix an element $\omega \in T$ once and for all, and let $k_\omega'$ be a closed subfield of $k_\omega$. For any $\nu \in T$ let $G_\nu$ be a $k_\nu$-algebraic group and let $G_\nu = G_\nu(k_\nu)$. Similarly let $G = \prod_{\nu \in T} G_\nu$, and $G = G(k_T)$.

Let $\mathbb{H}$ be an absolutely almost simple, connected, simply connected group defined over $k_\omega'$ which is $k_\omega'$-isotropic and suppose that $\mathbb{H}$ is a $k_\omega$-subgroup of $G_\omega$. Fix a $k_\omega'$-split torus $T$ of $\mathbb{H}$, and let $P$ be a minimal $k_\omega'$-parabolic subgroup of $\mathbb{H}$ which contains $T$. Let $U$ be the unipotent radical of $P$. Suppose $S$ is a one dimensional split subtorus of $T$, such that $Z_{\mathbb{H}}(U) \cap S = \{e\}$. Put $H = \mathbb{H}(k_\omega'), P = P(k_\omega'), S = S(k_\omega'),$ and $U = U(k_\omega')$.

Let $A$ be a locally compact second countable group and let $\Lambda$ be a discrete subgroup of $A$. Let $\mu$ be a Borel probability measure on $A/\Lambda$. Let $\Sigma$ be the closed subgroup of all elements of $A$ which preserve $\mu$. The measure $\mu$ is called homogeneous if there exists $x \in A/\Lambda$ such that $\Sigma x$ is closed and $\mu$ is the $\Sigma$-invariant probability on $\Sigma x$.

Let $G$ be a discrete subgroup of $G$ and put $X = G/T$. In this paper we prove

Theorem 1. If $\mu$ is a probability measure on $X$ which is invariant under the action of $SU$ and is $U$-ergodic, then $\mu$ is a homogeneous measure.

Let us now mention two corollaries of this theorem. It is not difficult to show that if $\mu$ is a probability measure on a measure space $X$ which is $SU$-invariant and mixing for the action of some $s \in S$, then $\mu$ is $U$-ergodic, see [R90a, Theorem ??]. Therefore, as a corollary of the above we get

Corollary 2. If $\mu$ is a probability measure on $X$ which is invariant under the action of $SU$ and is mixing for the action of some $s \in S$, then $\mu$ is a homogeneous measure.

Perhaps the most natural setting which gives rise to a measure satisfying conditions of Theorem 1 is when $\mu$ is an $H$-ergodic $H$-invariant measure on $X$. Indeed it follows from the Mautner phenomena, and the Howe-Moore Theorem that $\mu$ is $U$-ergodic. Therefore, as a corollary we have the following
Corollary 3. If $\mu$ is an $H$-invariant, $H$-ergodic measure on $X$, then $\mu$ is a homogeneous measure.

An interesting case to which the above statements apply is the arithmetic setting. In this case we can actually give more information about these measure, in some sense the group $\Sigma$ in the definition of homogeneous measure will have an algebraic description. The statement of that theorem involves definitions and notation which will be developed later, therefore, we postpone the statement and the proof of this refinement to section 6.

Let us mention that these statements are special cases of the celebrated measure classification theorem’s of Ratner in positive characteristic. Indeed in the case of local fields of characteristic zero these follows from Ratner’s measure classification theorem for the action of unipotent groups, see [R90b] and [R92] for the case of real Lie groups, and [MT94] and independently [R95] in the case of the product of real and $p$-aid groups. However, such results are not yet available in positive characteristic. It is plausible that if one assumes that the characteristic is “large”, then one can carry out the proof in [MT94] to this setting. Without such characteristic restrictions, however, this is far from being settled. In recent years there have been some partial results in this direction. The case of positive characteristic horospherical subgroups has been studied in [M08]. In [EM] a joining classification has been proved for the action of the maximal horospherical subgroup. The action of semisimple groups has also been considered, see [EG] where measure classification has been proved for the action of semisimple groups under the restriction that the characteristic is “large”.

Our results here remove the characteristic restriction from the main result in [EG]. It is worth mentioning that this generalization introduces serious technical difficulties. That is why, we are forced to use the ideas and techniques from [MT94] in an essential way, however, the proof in [EG] relies on a simpler argument which goes back to [E06]. We also borrow quite extensively from the theory of algebraic groups in this work. This also seems inevitable, since in this case there is no natural way to pass to the Lie algebra and read off the required data from there.

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2. Notation and preliminary lemmas

2.1. $k_T$-algebraic groups. Let $T$ be a finite set and let $k_\nu$ be a local field for all $\nu \in T$, and define $k_T = \prod_{\nu \in T} k_\nu$ as in the introduction. We endow $k_T$ with the norm $|\cdot| = \max_{\nu \in S} |\cdot|_\nu$, where $|\cdot|_\nu$ is a norm on $k_\nu$ for each $\nu \in T$.

A $k_T$-algebraic group $A$ (resp. variety $M$) is the formal product of $\prod_{\nu \in T} A_\nu$ of $k_\nu$ algebraic groups (resp. $\prod_{\nu \in T} M_\nu$ of $k_\nu$ algebraic varieties). The usual notions from elementary algebraic geometry theory e.g. regular maps, rational maps, rational point etc. are defined componentwise. We will take this as to be understood and use these notions without further remarks. There are two topologies on $M(k_T)$ the Hausdorff topology and the Zariski topology. When referring to the Zariski topology we will make this clear. Hence, if in a topological statement we do not give
reference to the particular topology used, then the one which is being considered is the Hausdorff topology.

Let $A$ be a $k_\nu$-algebraic group, and let $\lambda : G_m \to A$ be a noncentral homomorphism defined over $k_\nu$. Define $-\lambda(a) = \lambda(a)^{-1}$, for all $a \in k_\nu$. As in [Sp98], see also [CGP, Appendix C], we let $P_A(\lambda)$ denote the closed subgroup of $A$ formed by those elements $x \in A$, such that the map $\lambda(a)x\lambda(a)^{-1}$ extends to a map from $G_a$ into $A$.

Let $W^-_A(\lambda)$ be the normal subgroup of $P_A(\lambda)$, formed by $x \in P(\lambda)$ such that $\lambda(a)x\lambda(a)^{-1} \to e$ as $a$ goes to 0. The centralizer of the image of $\lambda$ is denoted by $Z_A(\lambda)$. Similarly define $W^+_A(\lambda)$ which we will denote by $W^+_A(\lambda)$.

Indeed $G_m$ acts on $\text{Lie}(A)$ via $\lambda$, and the weights are integers. The Lie algebras of $Z_A(\lambda)$ and $W^\pm_A(\lambda)$ maybe be identified with the weight subspaces of this action. It is shown in [Sp98], see also [CGP, Appendix C], which apparently is originally due to Borel and Tits [BT78], that $P_A(\lambda)$, $Z_A(\lambda)$ and $W^\pm_A(\lambda)$ are $k_\nu$ subgroups of $A$. Moreover, $W^\pm_A(\lambda)$ is a normal subgroup of $P_A(\lambda)$, and the product map

$$W^-_A(\lambda) \times Z_A(\lambda) \to P_A(\lambda)$$

is a $k_\nu$-isomorphism of varieties.

Furthermore, the product map

$$W^-_A(\lambda) \times Z_A(\lambda) \times W^+_A(\lambda) \to A$$

is a bijection onto a Zariski open subset of $A$. It is worth mentioning that these results are generalization to arbitrary groups of analogous and better known statements for reductive groups.

In the sequel we will work with $s = \lambda(a)$ for some $a \in k_\nu$ with $|a|_\nu > 1$, i.e. the weight subspaces are not all zero. Let $A = A(k_\nu)$, and let $W^+_A(s) = W^+_A(\lambda)(k_\nu)$ and $Z_A(s) = Z_A(\lambda)(k_\nu)$.

From (1) we conclude that $W^-_A(s)Z_A(s)W^+_A(s)$ is a Zariski dense subset of $A$, which contains a neighborhood of identity with respect to the Hausdorff topology.

We say an element $e \neq g \in A(k_T)$ is of class $A$ if $g = (g_\nu)_{\nu \in T}$ is diagonalizable over $k_T$, and for all $\nu \in T$ the component $g_\nu$ has eigenvalues which are integer powers of the uniformizer $\wp_\nu$ of $k_\nu$.

When working with algebraic groups over a non perfect field $k$, say with characteristic $p > 0$, it is convenient to use the language of group schemes. Indeed certain natural objects e.g. kernel of a $k$-morphism is not necessarily defined over the base field as linear algebraic groups in the sense of [B91] or [Sp98]. They are so called “$k$-closed”, i.e. they are closed group schemes over the base fields, however, they are not necessarily smooth group schemes. We have tried to avoid this language, and have tried to work in the more familiar setting of linear algebraic groups. Therefore, we will need to use the notation of a $k$-closed set, whose definition we now recall.

**Definition 4.** ([B91, AG, 12.1]) Let $A = \text{Spec}(K[x_1, \cdots, x_n]/I)$, where $I$ is the ideal of all functions vanishing on $A$. if the ideal $I$ is defined over $k$, the set $A$ is called $k$-closed.

Note that if $A$ is $k^p$-closed, then it is also $k$-closed. That is to say: $k$-topology and $k^p$-topology coincide. Let us mention that if $A \subset k^p$ is a set with is the zero set of an ideal $I$ in $k[x_1, \cdots, x_n]$, then $A$ is the $k$-points of a $k$-closed set. This
is how the $k$-closed sets will arise in our study. Abusing the notation a subset of $k^n$ will be called $k$-closed if it may be realized as the $k$-points of a $k$-closed subset of $\mathbb{A}^n$. A $k$-closed set is defined over $k$ as a Scheme but not as a variety. We have the following

**Lemma 5.** (cf. [CGP, Lemma D.3.1]) Let $\mathbb{A}$ be a scheme locally of finite type over a field $k$. There exists a unique geometrically reduced closed subscheme $\mathbb{A}' \subset \mathbb{A}$ such that $\mathbb{A}'(k') = \mathbb{A}(k')$ for all separable field extensions $k'/k$. The formation of $\mathbb{A}'$ is functorial in $\mathbb{A}$, and commute with the formation of products over $k$ and separable extension of the ground field. In particular, if $\mathbb{A}$ is a $k$-group scheme, the $\mathbb{A}'$ is a smooth $k$-subgroup scheme.

Let us also recall the definition of the Weil restriction of scalar.

**Definition 6.** Let $F$ be a field and $F'$ a subfield of $F$ such that $F/F'$ is a finite extension, and let $X$ be an affine $F$-variety. The Weil restriction of scalars $\mathcal{R}_{F/F'}(X)$ is the affine $F'$-scheme satisfying the following universal property

$$\mathcal{R}_{F/F'}(X)(A) = X(F \otimes_{F'} A)$$

for any $F'$-algebra $A$.

2.2. **Ergodic measures on algebraic varieties.** Let $\mathbb{A}$ be a $k_T$-algebraic group and let $A = \mathbb{A}(k_T)$. Let $\mathbb{B}$ be a connected $k_T$-subgroup of $\mathbb{A}$ so that $B = \mathbb{B}(k_T)$ is generated by one parameter $k_T$-split unipotent algebraic subgroups and $k_T$-split tori. Let $\Gamma$ be a discrete subgroup of $A$ and let $\pi : A \rightarrow A/\Gamma$ be the natural projection. For the proof of the following when $\mathbb{M}$ is a $k_T$-subvariety we refer to [MT94, Proposition 3.2] and [Sh99]. Thanks to Lemma 5, the proof goes though when this assumption is replaced by $k_T$-closed.

**Lemma 7.** Let $\mu$ be a $B$-invariant $B$-ergodic Borel probability measure on $A/\Gamma$. Suppose $\mathbb{M}$ is a $k_T$-closed subvariety of $\mathbb{A}$, and let $M = \mathbb{M}(k_T)$. If $\mu(\pi(M)) > 0$, then there exists a $k_T$-algebraic subgroup $D$ of $\mathbb{A}$ such that $B \subseteq D(k_T) = D$, and a point $x \in M$ such that $Dx \subset M$ and $\mu(\pi(Dx)) = 1$.

Let the notation be as in the beginning of 2.2. We will say a Borel probability measure $\mu$ on $A/\Gamma$ is Zariski dense if there is no proper $k_T$-closed subvariety $\mathbb{M}$ of $\mathbb{A}$ such that $\mu(\pi(M)) > 0$, where $M = \mathbb{M}(k_T)$. Two $k_T$-subvariety $L_1$ and $L_2$ of $\mathbb{A}$ are said to be transverse at $x$ if they both are smooth at $x$ and

$$T_x(L_1) \oplus T_x(L_2) = T_x(\mathbb{A}),$$

where $T_x(\bullet)$ denotes the tangent space of $\bullet$ at $x$. Thanks to the lemma 7, we also have the following from [MT94].

**Lemma 8.** Let the notation be as above, further, assume that $\mu$ is Zariski dense $B$-invariant Borel probability measure on $A/\Gamma$. Suppose $L$ is a connected $k_T$-algebraic subvariety of $\mathbb{A}$ containing $e$ which is transverse to $\mathbb{B}$ at $e$. Let $\mathbb{M}$ be a $k_T$-closed subvariety of $L$ containing $e$. There exists a constant $0 < c < 1$ such that if $\Omega \subset A/\Gamma$ is a measurable set with $\mu(\Omega) > 1 - c$, then one can find a sequence such that (i) $\{g_n\}$ converges to $e$, (ii) $g_n\Omega \cap \Omega \neq \emptyset$, and (iii) $\{g_n\} \subset L(k_T) \setminus M(k_T)$.
2.3. Homogeneous measures. Let $A$ be a locally compact second countable group, and let $\Lambda$ be a discrete subgroup of $A$. Suppose $\mu$ is a Borel probability measure on $A/\Lambda$. Let $\Sigma$ be the closed subgroup of all elements of $A$ which preserve $\mu$. The measure $\mu$ is called homogeneous if there exists $x \in A/\Lambda$ such that $\Sigma x$ is closed, and $\mu$ is the $\Sigma$-invariant probability measure on $\Sigma x$.

Lemma 9. (cf. [MT94, Lemma 10.1]) Let $A$ be a locally compact second countable group and $\Lambda$ a discrete subgroup of $A$. If $B$ is a normal unimodular subgroup of $A$, and $\mu$ is a $B$-invariant $B$-ergodic measure on $A/\Lambda$, then $\mu$ is homogeneous, moreover, $\Sigma = B\Lambda$.

2.4. Modulus of conjugation. If $A$ is a $k_T$-algebraic group, then $A = A(k_T)$ is a locally compact group. We let $\theta$ denote the Haar measure on $A$. For an element $a \in A$, we let $\alpha(a, A)$ denote the modulus of the conjugation action of $a$ on $A$, i.e. if $Y \subset A$ is a measurable set

$$\theta(aYa^{-1}) = \alpha(a, A)^{-1}\theta(Y).$$

Indeed $\alpha(a, A) = 1$ if $a \in [A, A]$, in particular, if $B$ is a subgroup of $A$ such that $[B, B] = B$, then $\alpha(b, A) = 1$ for all $b \in B$.

3. Algebraic statements

From a philosophical standpoint, one reason measure classification is particularly interesting and difficult is that: it connects measures, which are closely connected to the Hausdorff topology of the underlying group, to objects which are described using the Zariski topology, e.g. algebraic subgroups. In positive characteristic these two topologies are rather “far” from each other, e.g. there is no analogue of Hilbert’s fifth problem in general. This is one reason why the existing proofs in characteristic zero do not generalize easy to this case.

Let us restrict ourselves to unipotent groups to highlight some differences. In characteristic zero case the group generated by one unipotent matrix is already carries quite a lot of information, e.g. it is Zariski dense in a one dimensional group. In positive characteristic, however, all elements are torsion. The situation gets much better in the presence of a split torus action, in a sense, such action can be used to “define the Zariski closure” of the group generated by one element. This philosophy is used in this section, where we prove the following

**Proposition 10.** Let $K$ be a local field of characteristic $p > 0$ and $U$ a unipotent $K$-group equipped with a $K$-action by $GL_1$. Assume that all the weights are positive integers. Let $U$ be a subgroup $U(K)$, which is invariant under the action of $K^*$. Then there is $q$ a power of $p$ which only depends on the set of weights, such that $U = U'(K^q)$, where $U'$ is the Zariski-closure of $U \subseteq U(K) = R_{K/K^q}(U)(K^q)$ in $R_{K/K^q}(U)$.

Such groups arise naturally in our study, see Section 5 for more details.

We shall prove Proposition 10 in several steps. First we prove it when $U$ is a commutative $p$-torsion $K$-group. In the next step, the general commutative case is

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1See [P] where structure of compact subgroups of semisimple groups of adjoint type is described.
handled. In the final step, the general case is proved by induction on the nilpotency length.

In order to prove the first step, we shall start with a few auxiliary lemmas. In Lemmas 11 and 12, we assume that the weights are powers of \( p \).

Lemma 11. Let \( k \) be an infinite field of characteristic \( p \) and \( 0 < l_1 < \cdots < l_n \) positive integers. Assume \( \text{GL}_1 \) acts linearly on a standard \( k \)-vector group \( \mathbb{U} \) with weights equal to \( p^{l_i} \). Let \( \mathbb{U}_i \) be the weight space of \( p^{l_i} \). Let \( \mathbb{U} \) be a subgroup of \( \mathbb{U}(k) \) which is invariant under \( \text{GL}_1(k) \). If \( \mathbb{U} \) does not intersect \( \oplus_{i=2}^{n} \mathbb{U}_i \), then

\[
\mathbb{U} = \mathbb{U}'(k^{l_1}),
\]

for any \( l \geq l_1 \), where \( \mathbb{U}' \) is the Zariski-closure of \( \mathbb{U} \) in \( \mathbb{R}_{k/k^{l_1}}(\mathbb{U}) \).

Proof. Via the action of \( \text{GL}_1(k) \), one can view \( \mathbb{U}(k) \) as a \( k \)-vector space and \( \mathbb{U} \) as a \( k \)-subspace. Since \( \mathbb{U} \) does not intersect \( \oplus_{i=2}^{n} \mathbb{U}_i \), we get a \( k \)-linear map \( \theta \) from \( \text{pr}_1(U) \) to \( \oplus_{i=2}^{n} \mathbb{U}_i \), where \( \text{pr}_1 : \mathbb{U} \to \mathbb{U}_1 \) is the projection map, and we have

\[
\mathbb{U} = \{ (x, \theta(x)) \mid x \in \text{pr}_1(U) \}.
\]

It is clear that \( \text{pr}_1(U) \) is a \( k^{l_1} \)-subspace of \( \mathbb{U}_1(k) \) with respect to the standard scalar multiplication. It is also clear that \( \theta \) can be extended to a \( k \)-morphism from \( \mathbb{U}_1 \) to \( \oplus_{i=2}^{n} \mathbb{U}_i \). Hence there is a standard \( k^{l_1} \)-vector subgroup \( \mathbb{U}'_1 \) of \( \mathbb{R}_{k/k^{l_1}}(\mathbb{U}_1) \) such that

\[
\mathbb{U} = \{ (x, \theta(x)) \mid x \in \mathbb{U}'_1(k^{l_1}), y = \mathbb{R}_{k/k^{l_1}}(\theta)(x) \}.
\]

Since \( k \) is an infinite field, the Zariski-closure \( \mathbb{U}' \) of \( \mathbb{U} \) in \( \mathbb{R}_{k/k^{l_1}}(\mathbb{U}) \) is equal to

\[
\{ (x, y) \mid x \in \mathbb{U}'_1, y = \mathbb{R}_{k/k^{l_1}}(\theta)(x) \},
\]

which shows that \( \mathbb{U} = \mathbb{U}'(k^{l_1}) \). Now, one can easily deduce the same result for any \( l \geq l_1 \). \( \Box \)

Lemma 12. Let \( k \) be an infinite field of characteristic \( p \) and \( 0 < l_1 < \cdots < l_n \) positive integers. Assume \( \text{GL}_1 \) acts linearly on a standard \( k \)-vector group \( \mathbb{U} \) with weights equal to \( p^{l_i} \). Let \( \mathbb{U} \) be a subgroup of \( \mathbb{U}(k) \) which is invariant under \( \text{GL}_1(k) \). Then

\[
\mathbb{U} = \mathbb{U}'(k^{l_1}),
\]

for any \( l \geq l_n \), where \( \mathbb{U}' \) is the Zariski-closure of \( \mathbb{U} \) in \( \mathbb{R}_{k/k^{l_1}}(\mathbb{U}) \).

Proof. We denote the weight space of \( p^{l_i} \) by \( \mathbb{U}_i \). Let \( U_i = \mathbb{U} \cap (\oplus_{j=1}^{n} \mathbb{U}_j) \) and \( U'_i \) be a \( \text{GL}_1(k) \)-invariant complement of \( U_i \) in \( U_{i-1} \). So we have

\[
\begin{align*}
U &= U'_2 \oplus U'_3 \oplus \cdots \oplus U'_n \oplus U_n \\
U_{i-1} &= U_i \oplus U'_i \\
U'_i &\subseteq \bigoplus_{j=i-1}^{n} \mathbb{U}_j \\
U'_i \cap (\oplus_{j=i}^{n} \mathbb{U}_j) &= \{0\}
\end{align*}
\]
By Lemma 11, we have that \( U'_l = U'_l(\kappa^{p^l}) \), for any \( i \) and \( l \geq l_{i-1} \), where \( U'_l \) is the Zariski-closure of \( U'_l \) in \( R_{K/\kappa^{p^l}}(\oplus_{j=i-1}^{\infty} U_j) \). So one can easily finish the argument. \( \square \)

**Lemma 13.** Let \( m_1, \ldots, m_d \) be distinct positive integers which are coprime with \( p \). Let \( f(x) = (x^{m_1}, \ldots, x^{m_d}) \) be a morphism from \( \mathbb{A}^1 \) to \( \mathbb{A}^d \). Then
\[
F(x_1, \ldots, x_d) := f(x_1) + \cdots + f(x_d)
\]
is a separable function from \( \mathbb{A}^d \) to \( \mathbb{A}^d \) at a \( k \)-point, for any characteristic \( p \) infinite field \( k \).

**Proof.** It is enough to show that \( dF \) is invertible at some \( k \)-point. Since all of \( m_i \) are coprime with \( p \), it is enough to show the kernel of \( D = [x_1^{m_1-1}] \) is trivial, for some \( x_1 \in k \). Since \( k \) is an infinite field, it has an element \( x \in k \) of multiplicative order larger than \( \max_i m_i \). Set \( x_1 = x^{i-1} \). If \( D \) has a non-trivial kernel, then there is non-zero polynomial \( Q \) of degree at most \( d - 1 \) with coefficients in \( k \), such that
\[
Q(x^{m_1-1}) = Q(x^{m_2-1}) = \cdots = Q(x^{m_d-1}) = 0.
\]
This is a contradiction as \( x^{m_i-1} \) are distinct and the degree of \( Q \) is at most \( d - 1 \). \( \square \)

**Lemma 14.** Let \( K \) be a local field of characteristic \( p \) and \( m_1, \ldots, m_d \) distinct positive integers which are coprime with \( p \). Let \( U \) be a standard \( K \)-vector group equipped with a linear \( K \)-action by \( \text{GL}_1 \). Assume that the set of weights \( \Phi = \cup_{l=1}^{n} \Phi_l, \Phi_l = \{ p^l m_i \in \Phi \} l \in \mathbb{N} \}, and moreover \( \Phi_l \) is non-empty. Let \( U \) be a subgroup of \( U(K) \), which is invariant under the action of \( \text{GL}_1(K) \). Then
\[
U = U_1 \oplus \cdots \oplus U_d,
\]
where \( U_m \) is the weight space of \( m \). Furthermore, if \( \Phi_l = \{ p^l m_1, p^{2l} m_i, \ldots, p^{nl} m_i \} \)
and \( x = (x_1, \ldots, x_n) \in U_l \), then \( (\lambda^{p^l} x_1, \ldots, \lambda^{p^{nl}} x_n) \in U_l \), for any \( \lambda \in K^\times \).

**Proof.** Take an arbitrary element \( x = (x_m)_{m \in \Phi} \in U \). Since \( U \) is invariant under the action of \( \text{GL}_1(K) \), \((\lambda^m x_m)_{m \in \Phi} \) is also in \( U \), for any \( \lambda \in K^\times \). On the other hand, as \( U \) is a group,
\[
((\lambda_1^m + \lambda_2^m + \cdots + \lambda_d^m) x_m)_{m \in \Phi} \in U,
\]
for any \( \lambda_1, \ldots, \lambda_d \in K^\times \). On the other hand, by Lemma 13 and the inverse function theorem, for any \( \lambda'_1, \ldots, \lambda'_d \in K \), one can find \( \lambda_i \in K \) such that
\[
\lambda_i = \lambda_1^m + \lambda_2^m + \cdots + \lambda_d^m,
\]
for any \( 1 \leq i \leq d \). Using Equations 9 and 10, one can easily finish the argument. \( \square \)

**Lemma 15.** Let \( K \) be a local field of characteristic \( p \). Let \( U \) be a \( p \)-torsion commutative unipotent \( K \)-group equipped with a linear \( K \)-action by \( \text{GL}_1 \). Assume that all the weights are positive. Let \( U \) be a subgroup of \( U(K) \), which is invariant under the action of \( \text{GL}_1(K) \). Then there is \( l_0 \) depending only on the weights such that for any integer \( l \geq l_0 \)
\[
U = U'(K^{p^l}),
\]
where \( U' \) is the Zariski-closure of \( U \) in \( R_{K/K^{p^l}}(U) \).
Proof. By [CGP, Proposition B.4.2], we can assume that \( U \) is a \( K \)-vector group equipped with a \( K \)-linear \( GL_1 \) action. By Lemma 14, we can decompose \( U \) into subgroups \( U_i \) and get a new \( GL_1 \) action on \( \oplus_{m \in \Phi} U_m \) such that all the new weights are powers of \( p \) and \( U_i \) is invariant under the new action of \( GL_1(K) \). So we can finish the proof using Lemma 12. \( \square \)

**Lemma 16.** Let \( K \) be a local field of characteristic \( p \). Let \( U \) be a commutative unipotent \( K \)-group equipped with a \( K \)-action by \( GL_1 \) such that \( Z_{GL_1}(U) = \{1\} \). Let \( U \) be a subgroup of \( U(K) \), which is invariant under the action of \( GL_1(K) \). Then there is \( l_0 \) depending only on the weights of the action of \( GL_1 \) on Lie\( (U) \) such that for any integer \( l \geq l_0 \)

\[
U = U'(K^l),
\]

where \( U' \) is the Zariski-closure of \( U \) in \( R_{K/K^l}(U) \).

**Proof.** Since \( U \) is unipotent, it is a torsion group. Now we proceed by induction on the exponent of \( U \). If it is \( p \), by Lemma 15, we are done. Now assume that the exponent of \( U \) is \( p^i \). Let \( U[p] = \{ u \in U | u^p = 1 \} \). Since \( U \) is commutative, \( U[p] \) is a subgroup of \( U \) which is clearly invariant under the action of \( GL_1(K) \). Let \( V \) be its Zariski-closure in \( U \). Thus \( V \) is a \( p \)-torsion commutative unipotent \( K \)-group. Hence, by Lemma 15 for a large enough power of \( p \) (depending only on the weights), \( U[p] = U^{(p)}(K^q) \), where \( U^{(p)} \) is the Zariski-closure of \( U[p] \) in \( R_{K/K^l}(U) \). Let \( U'' \) be the Zariski-closure of \( U \) in \( R_{K/K^l}(U) \). Notice that \( GL_1 \) acts on \( U'' \) with no trivial weights and \( U^{(p)} \) is invariant under this action. Hence both of these groups and their quotient are \( K^q \)-split unipotent groups. We consider the following exact sequence of \( K^q \)-split unipotent groups,

\[
1 \to U^{(p)} \to U'' \xrightarrow{\pi} U''/U^{(p)} \to 1
\]

Notice that \( \pi(U) \) is Zariski-dense in \( U''/U^{(p)} \) and \( p^{-1} \)-torsion. Thus \( U''/U^{(p)} \) is \( p^{-1} \)-torsion. Hence, by induction hypothesis, for large enough power of \( p \) (depending only on the weights), we have that

\[
\pi(U) = U'(K^q),
\]

where \( U' \) is the Zariski-closure of \( \pi(U) \) in \( R_{K^q/K^q}(U''/U^{(p)}) \). On the other hand,

\[
U[p] = U^{(p)}(K^q) = R_{K^q/K^q}(U^{(p)})(K^q)
\]

and, by [Oe84, Corollary A.3.5], \( R_{K^q/K^q}(U^{(p)}) \) is \( K^q \)-split unipotent group. Thus \( U[p] \) is Zariski-dense in \( R_{K^q/K^q}(U^{(p)}) \). By [Oe84, Proposition A.3.8], we also know that

\[
1 \to R_{K^q/K^q}(U^{(p)}) \to R_{K^q/K^q}(U'') \xrightarrow{\pi'} R_{K^q/K^q}(U''/U^{(p)}) \to 1.
\]

Now let \( U' \) be the Zariski-closure of \( U \) in \( R_{K/K^l}(U') \). By the above discussion, it is easy to get the following short exact sequence and show that all of the involved groups are \( K^q \)-split unipotent groups,

\[
1 \to R_{K^q/K^q}(U^{(p)}) \to U' \xrightarrow{\pi} U' \to 1.
\]
By (11) and (12) and the fact that these groups are $K^q$-split unipotent groups, we get the following exact sequence,

\[ 1 \to U[p] \to \mathbb{U}'(K^q) \xrightarrow{\pi} \pi(U) \to 1. \tag{13} \]

So, by (13) and $U \subseteq \mathbb{U}'(K^q)$, one can easily deduce that $U = \mathbb{U}'(K^q)$, which finishes the proof. \hfill \Box

**Proof of Proposition 10.** We proceed by induction on the nilpotency length of $\mathbb{U}$. If it is commutative, by Lemma 16, we are done. Assume $\mathbb{U}$ is of nilpotency length $c$. Then $[U, U] \subseteq D(U)(K)$, where $D(U)$ is the derived subgroup of $U$ and $[U, U]$ is the derived subgroup of $U$. The nilpotency length of $D(U)$ is $c - 1$, and so, by induction hypothesis, we have that, for large enough power of $p$ (depending only on the weights),

\[ [U, U] = \tilde{\mathbb{U}}(K^q), \]

where $\tilde{\mathbb{U}}$ is the Zariski-closure of $[U, U]$ in $R_{K/K^q}(\mathbb{U})$. Let $\mathbb{U}''$ be the Zariski-closure of $U$ in $R_{K/K^q}(\mathbb{U})$. We notice that as $\text{GL}_1$ acts on $\mathbb{U}''$ with no trivial weight and $\tilde{\mathbb{U}}$ is invariant under this action, both of these groups and the quotient group are $K^q'$-split groups. We consider the following short exact sequence

\[ 1 \to \tilde{\mathbb{U}} \to \mathbb{U}'' \xrightarrow{\pi''} \mathbb{U}''/\tilde{\mathbb{U}} \to 1. \tag{14} \]

Since $\pi(U)$ is commutative and Zariski-dense in $\mathbb{U}''/\tilde{\mathbb{U}}$, we have that $\mathbb{U}''/\tilde{\mathbb{U}}$ is commutative. Therefore, by Lemma 16, for large enough power $p$ (depending only on the weights), we have

\[ \pi(U) = \mathbb{U}'(K^q), \tag{15} \]

where $\mathbb{U}'$ is the Zariski-closure of $\pi(U)$ in $R_{K'/K^q}(\mathbb{U})$. We also have

\[ [U, U] = \tilde{\mathbb{U}}(K^q) = R_{K'/K^q}(\mathbb{U}'')(K^q). \tag{16} \]

By [Oe84, Proposition A.3.8] and (14), we have

\[ 1 \to R_{K'/K^q}(\tilde{\mathbb{U}}) \to R_{K'/K^q}(\mathbb{U}'') \xrightarrow{\pi'} R_{K'/K^q}(\mathbb{U}''/\tilde{\mathbb{U}}) \to 1. \tag{17} \]

Let $\mathbb{U}'$ be the Zariski-closure of $U$ in $R_{K'/K^q}(\mathbb{U}'')$. Since $\tilde{\mathbb{U}}$ is a $K^q'$-split unipotent group, by (15) and (17), we have the following exact sequence

\[ 1 \to R_{K'/K^q}(\tilde{\mathbb{U}}) \to \mathbb{U}' \xrightarrow{\pi'} \mathbb{U}' \to 1 \]

and so, by (16), (17) and the fact that all the involved groups are $K^q$-split unipotent groups, we have

\[ 1 \to [U, U] \to \mathbb{U}'(K^q) \to \pi(U) \to 1, \tag{18} \]

and one can easily finish the proof. \hfill \Box

As a corollary of Proposition 10 we get the following

**Corollary 17.** (cf. [BS68, Proposition 9.13]) With the notation as above there exists a $k^\times$-invariant cross-section for $\mathbb{U}$ in $W$.  

Proof. It follows from Proposition 10 and its proof that; there exits a connected 
k^{r'}-algebraic subgroup \( U' \) of \( \mathbb{W}' = \mathcal{R}_{k/k}^R \mathbb{W} \) such that \( U = U'(k^r) \). Moreover \( U' \) is 
invariant under the action of \( S \), where \( S \) is the \( k^r \)-split torus obtained by taking the 
Zariski closure of \( k^x \) in \( \mathcal{R}_{k/k}^R(\mathbb{G}_m) \). Now [BS68, Proposition 9.13] implies \( U' \) has an \( S \)-equivariant \( k^r \)-isomorphism of \( k^r \)-varieties \( \phi: U' \times (\mathbb{W}'/U') \to \mathbb{W}' \). Let \( V = \phi((\mathbb{W}'/U')(k^r)) \). Then \( V \) is a \( k^x \)-invariant cross-section for \( U'(k^r) = U \) in \( \mathbb{W}'(k^r) = W \). \( \square \)

4. POLYNOMIAL LIKE BEHAVIOR AND THE BASIC LEMMA

Let \( \mu \) be a probability measure on \( X \) which is invariant and ergodic under the action 
of some unipotent \( k_T \)-algebraic subgroup of \( G \). In this section we will recall an 
important construction, which shows how polynomial like behavior of the action of 
unipotent groups on \( X \) can be used to acquire new elements which leave \( \mu \) invariant.
The idea of using polynomial like behavior of unipotent groups in the “intermediate 
range” dates back to several important works, e.g. Margulis’ celebrated proof of 
Oppenheim’s conjecture [Mar86], using topological arguments, and Ratner’s seminal 
work on the proof of measure rigidity conjecture [R90a, R90b, R91]. We keep 
the language of [MT94], as it is the most suitable one in our situation.

4.1. Construction of quasi-regular maps. Following [MT94, Section 5], we 
want to construct quasi-regular maps. This section is written in a more general 
setting and needed for the proof of Theorem 1, namely \( \mu \) is not assumed to be 
ergodic for the action of the unipotent group \( U \) below. Let us first recall the 
definition of a quasi-regular map. Here the definition is given in the case of a local 
field, which is what we need later, the \( T \)-arithmetic version is a simple modification.

Definition 18. (cf. [MT94, Definition 5.3]) Let \( \nu \) be any place in \( T \).
(i) Let \( E \) be a \( k_\nu \)-algebraic group, \( \mathcal{W} \) a \( k_\nu \)-algebraic subgroup of \( E(k_\nu) \), and \( \mathcal{M} \) a \( k_\nu \)-algebraic variety. A \( k_\nu \)-rational map \( f: \mathcal{M}(k_\nu) \to E(k_\nu) \) is called \( \mathcal{W} \)-quasiregular if the map from \( \mathcal{M}(k_\nu) \) to \( \mathcal{W} \), given by \( x \to \rho(f(x))p \) is \( k_\nu \)-regular for every \( k_\nu \)-rational representation \( \rho: E \to \text{GL}(\mathcal{W}) \), and every point \( p \in \mathcal{W}(k_\nu) \) such that \( \rho(\mathcal{W})p = p \).
(ii) If \( E = E(k_\nu) \) and \( \mathcal{W} \subset E \) is a \( k_\nu \)-split unipotent subgroup, a map \( \phi: \mathcal{W} \to E \) is called strongly \( \mathcal{W} \)-quasiregular if there exist 
(a) a sequence \( g_n \in E \) such that \( g_n \to e \),
(b) a sequence \( \{\alpha_n : \mathcal{W} \to \mathcal{W}\} \) of \( k_\nu \)-regular maps of bounded degree,
(c) a sequence \( \{\beta_n : \mathcal{W} \to \mathcal{W}\} \) of \( k_\nu \)-rational maps of bounded degree,
(d) a Zariski open nonempty subset \( \mathcal{X} \subset \mathcal{W} \)
such that \( \phi(u) = \lim_{n \to \infty} \alpha_n(u)g_n\beta_n(u) \), and the convergence is uniform on the 
compact subsets of \( \mathcal{X} \).

We note that if \( \phi \) is strongly \( \mathcal{W} \)-quasiregular, then it indeed is \( \mathcal{W} \)-quasiregular. To see 
this, let \( \rho: E \to \text{GL}(\mathcal{W}) \) be a \( k_\nu \)-rational representation, and let \( w \in \mathcal{W} \) be a 
\( \mathcal{W} \)-fixed vector. For any \( u \in \mathcal{X} \) we have

\[
\rho(\phi(u))w = \lim_{n \to \infty} \rho(\alpha_n(u)g_n)w.
\]
uniformly bounded on compact sets of $\mathcal{X}$. Therefore, it converges to a polynomial map with coefficients in $k_\nu$. We thus conclude $\phi$ is $\mathcal{W}$-quasiregular.

We now want to construct quasi-regular maps in the setting of theorem 1. Let us recall that $G$ is a $k_T$-algebraic group, and $H$ is a $k'_\nu$-group where the inclusion into $G_\omega$ defined over $k_\nu$. Therefore, if we replace $G_\omega$ by $G'_\omega = R_{k_\omega/k_\nu}(G_\omega)$, we get a $k'_\nu$-algebraic group, $G'_\nu$, such that $G'_\nu(k'_\nu) = G(k_\omega)$. Furthermore, it follows from the universal property of Weil’s restriction of scalars that $H$ is a $k'_\nu$-algebraic subgroup of $G'_\nu$. Hence we may, and we will, assume that $k_\omega = k'_\nu$, and $G_\omega = G'_\omega$. For simplicity in notation, for the rest of this section we will denote $k = k_\omega$.

Recall from the introduction that we fixed a $k$-split torus $T$ of $H$ and set $T = T(k)$. Let $P$ be a minimal $k$-parabolic subgroup of $H$ which contains $T$, and let $U$ be the unipotent radical of $P$. Set $P = P(k)$ and $U = U(k)$.

Also recall that $S$ is one dimensional split subtorus of $T$ such that $Z_H(U) \cap S = \{e\}$, and put $S = S(k)$. Let $s \in S$ be such that $s$ expands all the roots in $U$.

Let $U \subset W^+_G(s)$ be a closed subgroup normalized $S$. Then by Proposition 10 we have $U$ is an "algebraic group". Let us recall the precise statement: there is $q = p^n$, depending on $U$ and the action of $S$ on $W_{G_\omega}(s)$, such that $U$ is the $k^q$-points of a connected $k^q$-algebraic unipotent subgroup of $R_{k/k^q}(W_{G_\omega}(s))$, and hence an algebraic subgroup of $R_{k/k^q}(G_\omega)$.

In the construction below we replace $k$ by $k^q$ in $k_T$, and abusing the notation we will still show this by $k_T$. We also replace $G_\omega$ with $R_{k/k^q}(G_\omega)$. This way, $U$ is an algebraic subgroup of $G_\omega$. Furthermore, we now replace $S$ by the Zariski closure of $k^\times$ in $R_{k/k^q}(G_m)$, which will be called $S'$. Note that $S'(k^q) = S$.

Let $V$ be an $S$-invariant cross-section for $U$ in $W^+_G(s)$, which exists thanks to Corollary 17. Let $L = W^-_G(s)Z_G(s)V$ be a rational cross section for $U$ in $G$.

We fix relatively compact neighborhoods $\mathcal{B}^+$ and $\mathcal{B}^-$ of $e$ in $W^+_G(s)$ and $W^-_G(s)$ respectively with the property that

$$\mathcal{B}^+ \subset s\mathcal{B}^+s^{-1}, \text{ and } \mathcal{B}^- \subset s^{-1}\mathcal{B}^-s.$$ 

Define a filtration in $W^+_G(s)$ and $W^-_G(s)$ as follows

$$\mathcal{B}^+_n = s^n\mathcal{B}^+s^{-n}, \text{ and } \mathcal{B}^-_n = s^{-n}\mathcal{B}^-s^n.$$ 

For any integer $n$, we set $U_n = \mathcal{B}^+_n \cap U$. Define $\ell^\pm : W^\pm(s) \to \mathbb{Z} \cup \{-\infty\}$, by

(i) $\ell^+(x) = j$ iff $x \in \mathcal{B}^+_j \setminus \mathcal{B}^+_j$, and $\ell^+(e) = -\infty$,

(ii) $\ell^-(x) = j$ iff $x \in \mathcal{B}^-_j \setminus \mathcal{B}^-_{j-1}$, and $\ell^-(e) = -\infty$.

Let $\{g_n\}$ be a sequence in $L U \setminus N_G(U)$ with $g_n \to e$. Since $L$ is a rational cross-section for $U$ in $G$, we get rational morphisms

$$\tilde{\phi}_n : U \to L, \text{ and } \omega_n : U \to U$$

such that $u g_n = \tilde{\phi}_n(u) \omega_n(u)$ holds for all $u$ in a Zariski open dense subset of $U$. 

Thanks to the fact $\mathcal{W}$ is split we can identify $\mathcal{W}$ with an affine space. Then

$$\psi_n : \mathcal{W} \to \mathcal{W} \text{ given by } \psi_n(u) = \rho(\alpha_n(u)g_n)w$$

is a sequence of polynomial maps of bounded degree. Moreover, this family is uniformly bounded on compact sets of $\mathcal{X}$. Therefore, it converges to a polynomial map with coefficients in $k_\nu$. We thus conclude $\phi$ is $\mathcal{W}$-quasiregular.
Recall that by a theorem of Chevalley, there exists a $k_T$-rational representation
\[ \rho : G \to \GL(V) \] and $v \in V$ such that
\[ (20) \quad \mathcal{U} = \{ g \in G : \rho(g)v = v \}. \]
According to this description we also have
\[ (21) \quad \rho(N_G(\mathcal{U}))v = \{ \rho(g)v \in \rho(G)v : \rho(\mathcal{U})x = x \}. \]
Fix a bounded neighborhood $\mathcal{B}(v)$ of $v$ in the vector space $V$ such that
\[ (22) \quad \rho(G)v \cap \mathcal{B}(v) = \rho(G)v \cap \mathcal{B}(v), \]
where the closure is taken with respect to the Hausdorff topology of $V$. Recall that $g_n \not\in N_G(\mathcal{U})$, thus there is a sequence of integers \{\(b(n)\)\} such that $b(n) \to \infty$ and $\rho(\mathcal{U}(n)+1g_n)v \not\in \mathcal{B}(v)$, and $\rho(\mathcal{U}_n g_n)v \subset \mathcal{B}(v)$ for all $m \leq b(n)$.
Define $k^q$-regular isomorphisms $\tau_n : \mathcal{U} \to \mathcal{U}$ as follows: for every $u \in \mathcal{U}$
\[ (23) \quad \lambda_n(u) = s^n u s^{-n} \quad \text{and set} \quad \tau_n = \lambda_{b(n)} \]
For any $n \in \mathbb{N}$, we now define the $k^q$-rational map $\phi_n : \mathcal{U} \to L$ by $\phi_n = \tilde{\phi}_n \circ \tau_n$. Let $\rho_L$ be the restriction to $L$ of the orbit map $g \mapsto \rho(g)v$, and define
\[ (24) \quad \phi'_n = \rho_L \circ \phi_n : \mathcal{U} \to V \]
It follows from the definition of $b(n)$ that $\phi'_n(\mathcal{B}_0) \subset \mathcal{B}(v)$, however, $\phi'_n(\mathcal{B}_1) \not\subset \mathcal{B}(v)$.
Note that $\phi'_n(u) = \rho(\alpha_n(u)g_n)v$, hence $\phi'_n : \mathcal{U} \to V$ is a $k^q$-regular morphism. Also, since $\mathcal{U}$ is a connected $k^q$-group, which is normalized by $S$, and since $Z_G(S) \cap \mathcal{U} = \{e\}$, we get from [BS68, Corollary 9.12] that $\mathcal{U}$ and its Lie algebra are $S$-equivariantly isomorphic as $k^q$-varieties. Hence $\{\phi'_n\}$ is a sequence of equicontinuous polynomials of bounded degree. Therefore, after possibly passing to a subsequence we assume that there exists a $k^q$-regular morphism $\phi' : \mathcal{U} \to V$ such that
\[ (25) \quad \phi'(u) = \lim_{n \to \infty} \phi'_n(u) \quad \text{for every} \quad u \in \mathcal{U}. \]
The map $\phi'$ is non-constant since $\phi'(\mathcal{B}_1) \not\subset \mathcal{B}(v)^c$, also since $g_n \to e$ we have $\phi'_n(e) \to v$ which is to say $\phi'(e) = v$.
Let $\mathcal{M} = \rho(L)v$, since $L$ is a rational cross-section for $\mathcal{U}$ in $G$ containing $e$, we have $\mathcal{M}$ is a Zariski open dense subset of $\rho(G)v$ and $v \in \mathcal{M}$. Let now $\phi : \mathcal{U} \to L$ be the $k_T$-rational morphism defined by
\[ (26) \quad \phi = \rho_L^{-1} \circ \phi'. \]
It follows from the construction that $\phi(e) = e$, and that $\phi$ is non-constant.

Claim. The map $\phi$ constructed above is strongly $\mathcal{U}$-quasiregular.
To see the claim note that by the above construction we have
\[ (27) \quad \phi(u) = \lim_{n \to \infty} \phi_n(u), \quad \text{for all} \quad u \in \phi^{-1}(\mathcal{M}) \]
Therefore, since the convergence in (25) is uniform on compact subsets, and since $\rho_L^{-1}$ is continuous on compact subsets of $\mathcal{M}$, we get that the convergence in (28) is also uniform on compact subsets of $\phi^{-1}(\mathcal{M})$. Recall that $\tau_n(u)g_n = \phi_n(u)\omega_n(\tau_n(u))$.
Hence for $u \in \phi^{-1}(\mathcal{M})$ we can write
\[ (28) \quad \phi(u) = \lim_{n \to \infty} \tau_n(u)g_n(\omega_n(\tau_n(u)))^{-1}, \]
which establishes the claim.

As we mentioned this construction is quite essential to our proof. We will need some properties of the map $\phi$ constructed above. The proofs of these facts are mutandis mutatis of the proofs in characteristic zero which are given in [MT94], and we will not reproduce the complete proofs in here.

**Proposition 19.** (cf. [MT94, 6.1 and 6.3]) The map $\phi$ maps $U$ into $N_G(U)$, furthermore there is no compact subset $C$ of $H$ such that $\text{Im}(\phi) \subset C U$.

**Proof.** We will use the notation as above. Recall from (21) that $N_H(U) = \{ h \in H : \rho(U)\rho(h)v = \rho(h)v \}$.

Thus we need to show that for any $u_0 \in U$ we have $\rho(u_0)\rho(\phi(u))v = \rho(\phi(u))v$ for all $u \in U$.

Let $u \in \phi^{-1}(M)$. We saw in (28) that $\phi(u) = \lim_{n \to \infty} \tau_n(u)g_n(\omega_n(\tau_n(u)))^{-1}$.

Therefore, we have $\rho(u_0\tau_n(u)g_n)v = \rho(\tau_n(\tau_n^{-1}(u_0)u)g_n)v$.

Note that $\tau_n^{-1}(u_0) \to e$ as $n \to \infty$, thus we have $\phi(u) \in N_H(U)$ for all $u \in \phi^{-1}(M)$.

The result now follows since $\phi^{-1}(M)$ is a Zariski dense subset of $U$.

To see the second assertion note that $\phi = \rho_L^{-1} \circ \phi'$, and $\phi'$ is a non-constant (hence unbounded) polynomial map. □

In the sequel we will construct the quasi-regular map $\phi$ as above using a sequence of elements $g_n \to \infty$ with the following property

**Definition 20.** (cf. [MT94, Definition 6.6]) A sequence $\{g_n\}$ is said to satisfy the condition $(\ast)$ with respect to $s$ if there exists a compact subset $C$ of $G$ such that for all $n \in \mathbb{N}$ we have $s^{-b(n)}g_n^{s^{b(n)}} \in C$.

We also recall the following

**Definition 21.** A sequence of measurable non-null sets $A_n \subset U$ is called an averaging net for the action of $U$ on $(X, \mu)$ if the following analogue of the Birkhoff pointwise ergodic theorem holds: For any continuous compactly supported function $f$ on $X$ and for almost all $x \in X$ one has

$$\lim_{n \to \infty} \frac{1}{\mu(A_n)} \int_{A_n} f(u)\,d\theta(u) = \int_X f(h)\,d\mu_{y(x)}(h).$$

As we mentioned in the beginning of this section, we will study $U$-orbits of two near by points in the intermediate range. That is: when the maps $\phi_n$ are close to their limit. The following states that these “small” pieces of orbits are already equidistributed. The lemma is not difficult and follows from standard facts, however, it serves as the dynamical counterpart to the above algebraic construction.

**Lemma 22.** (cf. [MT94, Section 7.2]) Let $A \subset U$ be relatively compact and non-null. Let $A_n = \lambda_n(A)$, then $\{A_n\}$ is an averaging net for the $U$ action on $(X, \mu)$. 

Our arguments use limiting procedures, such arguments cannot be carried out in measure theoretic sense. As usual one needs to take “good” subset of $X$, the following does the job.

**Definition 23.** $\Omega \subset X$ is said to be a set of uniform convergence relative to $\{A_n\}$ if for every $\varepsilon > 0$ and every continuous compactly supported function $f$ on $X$ one can find a positive number $N(\varepsilon, f)$ such that for every $x \in \Omega$ and $n > N(\varepsilon, f)$ one has

$$\left| \frac{1}{\mu(A_n)} \int_{A_n} f(ux) d\theta(u) - \int_X f(h) d\mu_{y(x)}(h) \right| < \varepsilon. \quad (30)$$

As is proved in [MT94, Section 7.3], it follows from Egoroff’s theorem and second countability of the spaces under consideration that: for any $\varepsilon > 0$ one can find a measurable set $\Omega$ with $\mu(\Omega) > 1 - \varepsilon$, which is a set of uniform convergence relative to $\{A_n = \lambda_n(A)\}$ for every relatively compact non-null subset $A$ of $U$.

The following is the main application of the construction of the quasi-regular maps. It provides us with the anticipated “extra invariance property”.

**Basic Lemma.** (cf. [MT94, Basic Lemma, 7.5]) Let $\Omega$ be a set of uniform convergence relative to averaging nets $\{A_n = \lambda_n(A)\}$ corresponding to arbitrary relatively compact non-null subset $A \subset U$. Let $\{x_n\}$ be a sequence of points in $\Omega$ with $x_n \to x \in \Omega$. Let $\{g_n\} \subset G \setminus N_G(U)$ be a sequence which satisfies condition (*) with respect to $s$. Assume further that $g_n x_n \in \Omega$ for every $n$. Suppose $\phi$ is the $U$-quasiregular map corresponding to $\{g_n\}$ constructed above, then the ergodic component $\mu_{y(x)}$ is invariant under $\text{Im}(\phi)$.

**Proof.** The same proof as in [MT94, Basic Lemma] with $U = U_0 = U$ and $p = id$ works here. \hfill \Box

## 5. Proof of the Theorem

Let the notation be as before. In particular $\mu$ is a probability measure on $X$ which is invariant and ergodic for the action of $H$. Let $U = U^+(s)$ be the maximal subgroup of $W^+_G(s)$ which leaves the measure $\mu$ invariant. Note that this a closed subgroup of $W^+_G(s)$ with respect to the Hausdorff topology, and since $\mu$ is $S$-invariant it is normalized by $S$. Hence using Proposition 10 we have: there exists $q$ depending on $U$ and the action of $S$ on $W^+_G(s)$ such that $U$ is the $k^q$-points of a connected $k^q$-algebraic subgroup of $R_{k/k_s}(G_w)$, where as before we denote $k = k_w$. Note that $R_{k/k_s}(G_w)$ is a $k'_T$-group where $k'_T = (k_w)^q \times \prod_{\nu \notin w} k_{\nu}$.

We need yet another important property of $\phi$. Let us begin by fixing some more notation, let $s \in S$ be as before. Following [MT94], define

$$\mathcal{F}(s) = \{g \in G : Ug \subset \overline{W^+_G(s)Z_G(s)U}^\mathbb{Z} \} \quad (31)$$

here the closure is the Zariski closure. Since $W^+_G(s)Z_G(s)$ is a subgroup of $G$ the above can be written as

$$\{g \in G : \overline{W^+_G(s)Z_G(s)U}^\mathbb{Z} \subset \overline{W^+_G(s)Z_G(s)U}^\mathbb{Z} \} \quad (32)$$
Thus the inclusion in (32) may be replaced by equality which implies

\[ \mathcal{F}(s) \text{ is a } k^\mathbb{A}\text{-closed subgroup of } G. \]

Note that \( S \subset \mathcal{F}(s) \), moreover, \( \mathcal{F}(s) \cap W_G^+(s) = \mathcal{U} \).

Put \( \mathcal{U}^- = \mathcal{F}(s) \cap W_G^-(s) \), this a \( k^\mathbb{A}\)-closed subgroup of \( W_G^-(s) \) which is normalized by \( S \). Let \( \mathcal{V}^- \) be an \( S \)-invariant cross-section for \( \mathcal{U}^- \). Existence of such cross-section follows from Corollary 17, in the case in hand however this can be directly deduced from [BS68, Proposition 9.13] combined with Lemma 5. It is worth mentioning that in general the smooth group Scheme obtained in Lemma 5 is not necessarily connected, however, in the case in hand because of the contracting action of \( S \) on \( \mathcal{U}^- \), it will be connected. Also recall that we fixed an \( S \)-invariant cross-section \( \mathcal{V} \) for \( \mathcal{U} \) in \( W_G^+(s) \).

As was mentioned in section 2.1

(33)

\[ \mathcal{D} = \mathcal{U}^- \mathcal{V}^- Z(s) \mathcal{V} \mathcal{U} = W_G^-(s) Z_G(s) W_G^+(s) \]

is a Zariski open dense subset of \( G \) and for any \( g \in \mathcal{D} \) we have a unique decomposition

(34)

\[ g = w^- (g) z (g) w^+ (g) = u^- (g) v^- (g) z (g) v (g) u (g) \]

where \( u^- (g) \in \mathcal{U}^- \), \( v^- (g) \in \mathcal{V}^- \), \( z (g) \in Z_G(s) \), \( u (g) \in \mathcal{U} \), \( v (g) \in \mathcal{V} \), \( w^- (g) = u^- (g) v^- (g) \), and \( w^+ (g) = v (g) u (g) \).

Note that for every \( w^\pm \in W_G^\pm (s) \) we have

(35)

\[ \ell^\pm (s^m w^\pm s^{-m}) = \ell^\pm (w^\pm (g)) \pm m \]

We need the following proposition from [MT94].

**Proposition 24.** (cf. [MT94, Proposition 6.7]) Suppose \( \{g_n\} \) is a sequence converging to \( e \), and let \( s \) and \( \mathcal{U} \) be as above. Suppose one of the following holds

(i) the sequence \( \ell^- (v^- (g_n)) - \ell^- (u^- (g_n)) \) is bounded from below,

(ii) \( \{g_n\} \subset Z_G(s) W_G^+ (s) \setminus N_G(\mathcal{U}) \),

then \( \{g_n\} \) satisfies condition (\( * \)), furthermore \( \phi (\mathcal{U}) \subseteq W_G^+ (s) \).

**Proof.** The fact that the conclusion holds under condition (i) is proved in [MT94, Proposition 6.7]. Therefore, let us assume (ii) holds, then \( s^{-b(n)} g_n s^{b(n)} \to e \) so \( \{g_n\} \) satisfies condition (\( * \)). We now show \( \phi (\mathcal{U}) \subseteq W_G^+ (s) \). The argument is the same as the argument given in [MT94] for the proof of the conclusion under assumption (i).

With notation as in Section 4, let \( u \in \phi^{-1} (\mathcal{M}) \). Then as we saw in (28) we have

\[ \varphi (u) = \lim_{n \to \infty} \tau_n (u) g_n \omega_n (\tau_n (u))^{-1} . \]

It follows from the definitions, in particular the choice of \( b(n) \), that \( \{s^{-b(n)} \bullet s^{b(n)}\} \) are bounded in \( \mathcal{U} \), for \( \bullet = \tau_n (u) \) and \( \omega_n (\tau_n (u))^{-1} \). After passing to a subsequence if needed, this implies

\[ \lim_{n \to \infty} s^{-b(n)} \varphi_n (u) s^{b(n)} = \lim_{n \to \infty} s^{-b(n)} \tau_n (u) g_n \omega_n (\tau_n (u))^{-1} s^{b(n)} \in \mathcal{U} . \]

Therefore, we get \( \varphi (u) \in W_G^+ (s) \). This, together with the fact that \( \phi^{-1} (\mathcal{M}) \) is Zariski dense in \( \mathcal{U} \), implies the claim. \( \square \)
The proof of this proposition goes through as in [MT94]. The following is an important consequence of the above proposition and the construction of quasi-regular maps in Section 4. It describes the local structure of the set of uniform convergence this is essential in the proof of measure rigidity by Ratner [R90b]. Our proposition however is taken from [MT94].

**Proposition 25.** (cf. [MT94, Proposition 8.3]) With the above notation we have: for every $\epsilon > 0$ there exists a compact subset $\Omega_\epsilon$ of $X$ with $\mu(\Omega_\epsilon) > 1 - \epsilon$ such that if $\{g_n\} \in G \setminus N_G(U^+(s))$ is a sequence with $g_n \to e$ and $g_n\Omega_\epsilon \cap \Omega_\epsilon \neq \emptyset$ for every $n$, then the sequence $\{\ell^-(v^-(g_n)) - \ell^-(u^-(g_n))\}$ tends to $-\infty$.

**Proof.** Note that $U \subset \mathcal{U}$, therefore, the measure $\mu$ is $\mathcal{U}$-ergodic invariant measure. Let $\epsilon > 0$ be given and let $\Omega_\epsilon$ be the set of uniform convergence for the action of $\mathcal{U}$ in the sense of the Definition 23 with $\mu(\Omega_\epsilon) > 1 - \epsilon$. We will show that $\Omega_\epsilon$ satisfies the conclusion of the proposition. Assume the contrary and let $\{g_n\}$ be a sequence for which the conclusion of the proposition fails for $\Omega_\epsilon$. Passing to a subsequence we may assume $\{\ell^-(v^-(g_n)) - \ell^-(u^-(g_n))\}$ is bounded from below. Therefore, Proposition ref;star guarantees that $\{g_n\}$ satisfies condition $(\ast)$. We now construct the quasi regular map $\phi$ corresponding to $\{g_n\}$ as in Section 4. By the basic lemma $\mu$ is invariant under $\text{Im}(\phi)$. Now by the Proposition ref;star we have $\text{Im}(\phi) \subset W^-_G(s)$, moreover, in view of Proposition 19 the image of $\phi$ is not contained in $C\mathcal{U}$ for any bounded set $C$. Hence, $\mu$ is invariant under $\langle \mathcal{U}, \text{Im}(\phi) \rangle$ which strictly contains $\mathcal{U}$, contradicting the maximality of $\mathcal{U}$. \hfill $\square$

We will use this proposition in the following form.

**Corollary 26.** (cf. [MT94, Corollary 8.4]) Let the notation be as before. There exists a subset $\Omega$ in $X$ with $\mu(\Omega) = 1$ such that $W^-_G(s)x \cap \Omega \subset \mathcal{U}^-x$, for every $x \in \Omega$.

**Proof.** The proof follows the same lines as the proof of Corollary 8.4, using Proposition 25. \hfill $\square$

We need certain properties of $F(s)$, which will be used in the application of entropy argument in the proof of Theorem 1. The main property we need is Lemma 27 below, which is a consequence of the fact that $U$ is expanded by $s$ and that $H$ is a semisimple group. In the course of the proof the fact that $F(s)$ is a $k^d$-closed subgroup is also used.

Let $F = (F(s))^\circ$, denote the connected component of the identity in the Zariski closure of $F(s)$. Then by Lemma 5, $F$ is a connected $k^d$-algebraic subgroup of $G$. Put $F = F(k^d)$, then $\mathcal{U} = F \cap W^+_G(s)$, $\mathcal{U}^- = F \cap W^-_G(s)$, and $S \subseteq F$. Furthermore, $F$ is Zariski dense in $F$ and we have

$$U^- \times (F \cap Z_G(s)) \cap \mathcal{U} \to F$$

is a diffeomorphism onto a Zariski open dense subset which contains the identity. Therefore we get

$$\text{Lie}(F) = \text{Lie}(\mathcal{U}^-) \oplus \text{Lie}(F \cap Z_G(s)) \oplus \text{Lie}(\mathcal{U}) \subseteq \text{Lie}(G).$$
Let us fix a norm, $\| \|$ on Lie($\mathbb{G}$). Let now $\Phi = \wedge^{\dim \mathbb{F}}\text{Lie}(\mathbb{G})$ and $\rho = \wedge^{\dim \mathbb{F}}\text{Ad}$. Further, let $q \in k \cdot \wedge^{\dim \mathbb{F}}\text{Lie}(\mathbb{F})$ be a unit vector. Then
\[ F \subseteq \{ g \in G : \rho(g)q \in k \cdot q \}, \]
and since $U$ is a unipotent subgroup $\rho(U)q = q$, in particular, it follows from $U \subseteq U$ that $\rho(U)q = q$.

We now prove the following lemma. The proof is taken from [Sh95, Lemma 5.2].

**Lemma 27.** Let the notation be as in section 2.1, then
\[ \alpha(s, U) \geq \alpha(s^{-1}, U^-) \tag{37} \]

**Proof.** Indeed this is equivalent to the fact that
\[ |\det(\text{Ad}(s))|_{\text{Lie}(U^-)\omega} \geq |\det(\text{Ad}(s^{-1}))|_{\text{Lie}(U^-)\omega}. \]
This, in turn, follows if we show $\|\rho(s)q\| \geq 1$. Assume the contrary, therefore
\[ \lim_{n \to \infty} \rho(s^n)q = 0. \]

Let $\Phi^-$ denote the subspace of $\Phi$ on which $\rho(s)$ acts by contraction, we now claim that this assumption implies $\rho(H)q \subseteq \Phi^-$. Let us assume this assertion and conclude the proof: indeed this implies $\Psi = \text{Span}(\rho(H)q) \subseteq \Phi^-$. Hence, we find a linear representation, $q$ say, of $H$ into GL($\Psi$) for which the $\det(\rho(s)) \neq 1$. This contradicts the fact that $H$ is a semisimple group.

In view of this discussion, we need to show the claim. Since $H$ is generated by $W_H^+(s) = U$ and $W_H^-(s)$, and since $q$ is fixed by $U$, we need to show $\rho(W_H^-(s)) \subseteq \Phi^-$. To see this, let $v \in W_H^-(s)$, then since $s^n vs^{-n} \to e$ as $n \to \infty$ we get
\[ \rho(s^n)\rho(v)q = \rho(s^n vs^{-n})\rho(s^n)q \to 0. \]

Thus $vq \in \Phi^-$ as we wanted to show. \[ \square \]

The following theorem is proved in [MT94]. The application of entropy is in a sense present but implicit in Ratner’s proof of measure rigidity, this has been made explicit more transparent in [MT94]. Suppose $s$ is an element from class $A$ which acts ergodically on the measure space $(X, \sigma)$, let $h(s, \sigma)$ denote the entropy of $s$. We have the following

**Theorem 28.** (cf. [MT94, Thm. 9.7])
Assume $s$ is an element from class $A$ which acts ergodically on the measure space $(X, \sigma)$. Let $U$ be an algebraic subgroup of $W^-_G(s)$ normalized by $s$. Put $\alpha = \alpha(s^{-1}, U)$.

(i) If $\sigma$ is $U$-invariant, then $h(s, \sigma) \geq \log_2 \alpha$.

(ii) Assume that there exists a subset $\Omega \subseteq X$ with $\sigma(\Omega) = 1$ such that for every $x \in \Omega$ we have $W^-_G(s)x \cap \Omega \subset Ux$. Then $h(s, \sigma) \leq \log_2 (\alpha)$ and the equality holds if and only if $\sigma$ is $U$-invariant.

**Proof of Theorem 1.** Let $\mu$ be as in the statement of Theorem 1. Note that $\mu$ is $s$-ergodic, indeed by Mautner phenomena any $s$-ergodic component is $U$-invariant which implies the claim since $\mu$ is $U$-ergodic. Let $U$ be as in the beginning of this section i.e. $U$ is the maximal subgroup of $W^+_G(s)$ which leaves $\mu$ invariant. We also let $k^q$ and $k^q_T$ be as in the beginning of this section. Recall that $G$ is realized as the $k^q_T$-points of a $k^q_T$-group. We now complete the proof in some steps.
Step 1. $\mu$ is invariant under $U^-$.  

By Corollary 25 there exists a full measure subset $\Omega \subset X$ such that  
\[ W_G^-(s) x \cap \Omega \subset U^- x \text{ for every } x \in \Omega. \]
Recall that $\mu$ is $s$-ergodic and that $h(s, \mu) = h(s^{-1}, \mu)$. We may now apply theorem 28 and get  
\[ \log_2 \alpha(s, U) \leq h(s, \mu) \leq \log_2 \alpha(s^{-1}, U^-) \]
Let us also note that  
\[ (s, \mathcal{F}(s)) = \alpha(s, U^-)^{-1} \alpha(s, U) \geq 1 \]
Note that by Theorem 28 we have  
\[ \log_2 \alpha(s, U) \leq h(s, \mu) \leq \log_2 \alpha(s^{-1}, U^-) \]
Thus equality holds, now another application of Theorem 28 implies that $\mu$ is invariant under $U^-$.  

Step 2. Reduction to Zariski dense measures.  

We now apply Lemma 7 with $B = \langle U^- , S, U \rangle$ and $A = G$. Hence we get a subgroup $G' \subseteq G$, and a point $x \in X$ such that $B \subseteq G'$ and $\mu(G'x) = 1$. Furthermore, there is no $k_T$-closed proper subvariety $M \subsetneq G'$ such that $\mu(\pi(M)) > 0$. Abusing the notation we let $\mathcal{V}$ (respectively $\mathcal{V}^-$) denote the cross-sections for $\mathcal{U}$ (resp. $\mathcal{U}^-$) in $W_{G'}(s)$ (resp. $W_{G'}^-(s)$).  

Step 3. $\mu$ is invariant under $W_{G'}^-(s)$.  

We will show $\mathcal{U}^- = W_{G'}^-(s)$. Assume the contrary, then, in particular, from the definition of $\mathcal{U}^-$ it follows that $\mathcal{U}$ is not a normal subgroup of $G'$. Therefore, since $\mu$ is Zariski dense it follows from Lemma 8 that: if $0 < \varepsilon < 1$ is small enough and $\Omega_\varepsilon$ is a measurable set with $\mu(\Omega_\varepsilon) > 1 - \varepsilon$, then there exists a sequence $\{g_n\}$ converging to $e$ such that  
\[ \{g_n\} \subset \mathcal{V}^- Z_{G'}(s) W_{G'}^+(s) \setminus \left( Z_{G'}(s) W_{G'}^+(s) \cup N_{G'}(\mathcal{U}) \right), \]
and $g_n \Omega_\varepsilon \cap \Omega_\varepsilon \neq \emptyset$ for all $n$.  

Hence we have $\ell^-(v(g_n)) > -\infty$ and $\ell^-(u^-(g_n)) = -\infty$, which contradicts Proposition 25. Therefore, $\mathcal{U}^- = W_{G'}^-(s)$ and thanks to the first step, we get $\mu$ is invariant under $W_{G'}^-(s)$.  

Step 4. Either $U$ is normal in $G'$ or $\mu$ is invariant under $W_{G'}^+(s)$.  

Let us assume $U$ is not normal in $G'$, then we need to show $U = W_{G'}^+(s)$. Assume the contrary, and let $\Omega_\varepsilon$ be as in Step 3. Another application of Zariski density of the measure, together with the facts that $\mu$ is invariant under $W_{G'}^-(s)$ and $\mathcal{U}$, implies that we can find  
\[ \{g_n\} \subset Z_{G'}(s) \mathcal{V} \setminus N_{G'}(\mathcal{U}), \text{ so that } g_n \Omega_\varepsilon \cap \Omega_\varepsilon \neq \emptyset, \text{ and } g_n \rightarrow e. \]
Thus for any $n$ we have $x_n \in \Omega_\varepsilon$ so that $y_n = g_n x_n \in \Omega_\varepsilon$. Construct the map $\phi$ using $\{g_n\}$. Now by Proposition 24 we get $\text{Im}(\phi) \subseteq W_{G'}^+(s) \setminus \mathcal{U}$. This contradicts the maximality of $U$. 

Therefore, we have proved that the measure \( \mu \) is invariant under \((W^+_G(s), W^-_G(s))\). This is a normal subgroup of \( G' \) which is generated by unipotent subgroups, and in particular it is unimodular. The theorem now follows from Lemma 9.

6. The arithmetic case

In this section we will give a refinement of Theorem 7 in the case \( \Gamma \) is an arithmetic lattice in \( G \). To be more explicit let \( K \) be a global field of characteristic \( p > 0 \). Let \( G \) be a connected, simply connected, almost simple group defined over \( K \). Let \( \mathcal{T} \) be a finite set of places of \( K \) and let \( K_{\mathcal{T}} = \prod_{\nu \in \mathcal{T}} K_{\nu} \). We let \( G = \prod_{\nu \in \mathcal{T}} G(K_{\nu}) \). Furthermore we will denote \( G_{\nu} = G(K_{\nu}) \). Recall that an arithmetic lattice compatible with the \( K \)-group \( G \) is a lattice commensurable with the subgroup of \( G \) consisting of all matrices (in a particular representation as a linear group) with entries in the ring of \( \mathcal{O} \)-integers which we will denote by \( \mathcal{O}_\mathcal{T} \).

Let \( S \) be a nonempty subset of \( \mathcal{T} \) and let \( l_{\nu} \) be a closed subgroup of \( K_{\nu} \) for all \( \nu \in S \). As is well known \( l_{\nu} \) is a local field and \( k_{\nu} \) is a finite extension of \( l_{\nu} \). We will let \( l_S = \prod_{\nu \in S} l_{\nu} \). For any \( \nu \in S \) let \( H_{\nu} \) be a connected, simply connected almost simple group defined over \( l_{\nu} \) and let \( H = \prod_{\nu \in S} H_{\nu} \).

Let us recall some notation from Section 2.1. Let \( \mathbb{A} \) be a group defined over a local field \( k \) and let \( A = \mathbb{A}(k) \). This is a locally compact group with respect to the topology induced from \( k \) which will be referred to as the Hausdorff topology in the sequel. Let \( \mathbb{P} \) be a pseudo-parabolic \( k \)-subgroup of \( \mathbb{A} \). The there is a noncentral \( k \)-homomorphism \( \lambda : GL_1 \to \mathbb{A} \) such that \( \mathbb{P} = \mathbb{P}(\lambda) \). Let \( W^+_\mathbb{P} = W(\lambda) \) be the unipotent radical of \( \mathbb{P} \) and let \( W^-_\mathbb{P} = W(-\lambda) \) and let \( W^+_A = W^+_\mathbb{P}(k) \). We let \( A^+_p = \langle W^+_A, W^-_A \rangle \), be the group generated by \( W^+_A \). This is a closed normal subgroup of \( A \) with respect to the Hausdorff topology hence if \( P \) and \( Q \) are two pseudo-parabolic which are conjugate in \( A \) then \( A^+_P = A^+_Q \) and hence using [CGP, Appendix C], in particular C.2.6, C.2.8 and C.2.10 in there, there are only finitely many subgroups \( A^+_P \).

We can now state our refinement of Theorem 7 in the arithmetic setting

**Theorem 29.** Let the notation be as in the beginning of this section. In particular \( \Gamma \) is an arithmetic lattice commensurable with \( \mathcal{O}(\mathcal{O}_\mathcal{T}) \) and \( G \) and \( H \) are as above. Let \( \mu \) be an \( H \)-invariant ergodic probability measure on \( G/\Gamma \). Then there exist

(i) \( k^+_\mathcal{T} = \prod_{\nu \in \mathcal{T}} k^+_{\nu} \subset K_{\mathcal{T}} \) where \( k^+_{\nu} \) is a closed subfield of \( K_{\nu} \). More precisely we have \( k^+_{\nu} = k_{\nu} \) if \( \nu \notin S \) and \( k^+_{\nu} = l^+_\nu \) for some \( q = p^n \) if \( \nu \in S \),

(ii) a connected \( K \)-subgroup \( \mathbb{F} \) of \( \mathcal{R}_{K_{\mathcal{T}}/k^+_\mathcal{T}}(G) \),

(iii) a point \( x = g\Gamma \in X \)

such that \( \mu \) is the unique probability Haar measure on the closed orbit \( \Sigma x \) where \( \Sigma = gF^+_pF_xg^{-1} \) and \( F = \mathbb{F}(k^+_\mathcal{T}) \), is \( F^+_p \) is defined as above, \( F_x \) is the stabilizer of \( x \) in \( F \) and the closure is with respect to the Hausdorff topology.

**Proof.** This theorem is essentially proved in the course of the proof of Theorem 7. We will however give the more detailed discussion here for the sake of completeness. We first show the statement for the case \( S \) is a singleton and \( \mathbb{H} \) is an absolutely almost simple group. In this case the proof of Theorem 7 gives the following. There is a \( k^+_{\mathcal{T}} \) as above a \( k^+_{\mathcal{T}} \)-connected subgroup \( \mathbb{F} \) such that \( \mathbb{F} = \mathbb{F}'(k^+_{\mathcal{T}}) \) contains
$H$ and a point $x = g\Gamma \in X$ such that $\mu(F'x) = 1$ and $\mu$ is invariant under \langle W_P^+(s), W_P^-(s) \rangle$. Using the above notation the later is of the form $F^+_P$, for some parabolic subgroup $P$. Let now $F'_x = F' \cap g\Gamma g^{-1}$ be the stabilizer of $x$. Hence lemma 9 implies $\mu$ is Haar measure on the closed orbit $\Sigma x$ where $\Sigma = F^+_P F^+_x$, the closure is with respect to the Hausdorff topology. Now the push forward of $\mu$ under the natural map $F'/F'_x \to F'/B$ gives an $H$-invariant ergodic measure on $(F'/B)(k'_F)$. In view of Lemma 7 we get $\mu = \mathbb{B} = \mathbb{B}(k'_F)$. This implies $g^{-1}F'_xg \cap \Gamma$ is Zariski dense in $g^{-1}F'g$. Hence $\mathbb{F} = g^{-1}F'g$ is defined over $K$ and this finishes the proof in the case $S$ is a singleton.

We now turn to the general case. Hence $H$ is as in the beginning of this section and $\mu$ is $H$-invariant and ergodic measure on $\Sigma x$. Fix $\omega \in S$ and let $H_\omega$ be a simple factor of $\mathbb{H}_\omega(l_\omega)$. Let $\mu = \int_Y \mu_y d\sigma$ be the ergodic decomposition of $\mu$ with respect to $H_\omega$. For $x \in X$ we let $y(x)$ denote the corresponding point from $(Y, \sigma)$. Since $\mu_y(x)$'s are $H_\omega$-ergodic the argument above gives: for almost all $y(x)$ the measure $\mu_y(x)$ is the invariant measure on a closed orbit of $\Sigma(x)$ where $\Sigma(x)$ is defined as above in particular there exist a $k'_F(x)$ as in (i) and a $k'_F$-group $F'$ of $R_{K_F/k^F}(\mathbb{G})$ such that $\mu(F'x) = 1$ where $F' = F'(k'_F)$. Furthermore $F' = F' g^{-1}$ where $F = F(k'_F)$ and $F$ is a $K$-subgroup of $R_{K_F/k^F}(\mathbb{G})$. We will refer to the pair $(k'_F, P, F)$ as a subgroup associated to $x$. For any triple $(k'_F, F, P)$ where $F$ is a $k'_F$-subgroup of $real_{K_F/k^F}(\mathbb{G})$ which also has a $K$-structure let $F = F(k'_F)$ and $P$ is a standard $k'_F$-pseudo parabolic of $F$. Define

$$\mathcal{S}(k'_F, F, P) = \{ x \in X : (k'_F, F, P) \text{ is associated to } x \}$$

Note that there are only countably many $k'_F$'s as in (i) and for each $k'_F$ there are only countably many such $F$ and for each $F$ there are only finitely many such $P$'s. Hence there exists a triple $(k'_F, F, P)$ such that $\sigma(\mathcal{S}(k'_F, F, P)) > 0$. Note also that $H$ normalizes $H_\omega$. So for every $h \in H$ the equality $\mu_{hy}(x) = \mu_{y(hx)}$ is true for every $\mu$-almost all $x \in X$, furthermore we have $\Sigma(hx) = h\Sigma(x)h^{-1}$. Hence $\mathcal{S}(k'_F, F, P)$ is $H$-invariant. This and the fact that $\mu$ is $H$-ergodic imply that $\sigma(\mathcal{S}(k'_F, F, P)) = 1$.

Observe that the map $x \mapsto \Sigma(x)^{-1}$, where the closure is the Zariski closure is a Borel map which is $H$-equivariant map from $\mathcal{S}(k'_F, F, P)$ to $G/N_G(F)$. Now we use lemma 7 and conclude that this map is constant almost everywhere. Hence $\mu$ is the $\Sigma$-invariant measure on a closed $\Sigma$-orbit where $\Sigma = gF^+_P F^+_x g^{-1}$ for some $g \in G$. This finishes the proof.

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