1. Prove that there are no integers m and n such that

$$7n + 21m = 51.$$

2. Prove that for any positive integer n we have

$$3^n \geq n+1.$$

[**Hint**: Use induction on n.]

3. (a) Let $\{f_n\}$ be the Fibonacci sequence, i.e.

$$f_0 = 0, \quad f_1 = 1 \quad \text{and} \quad f_{n+1} = f_n + f_{n-1}.$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Prove that for any positive integer n we have

$$A^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}.$$

(b) Use part (a) and the fact that $A^{m+n} = A^m \cdot A^n$ to prove that for any positive integers m and n we have:

$$f_{m+n} = f_{m-1} \cdot f_n + f_m \cdot f_{n+1}.$$

4. Let m and n be positive integers. Prove that, if $m \mid n$, then $f_m \mid f_n$.

[**Hint 1**: Using 3(b) and by induction on k, prove that $f_m \mid f_{mk}$ for any positive integer k.]

[**Hint 2**: $f_{m(k+1)} = f_{m+mk}$.]

[**Hint 3**: You can use the following fact without proof:

$d \mid \ell_1$ and $d \mid \ell_2 \Rightarrow d \mid s_1\ell_1 + s_2\ell_2$ for any integers s_1, s_2.

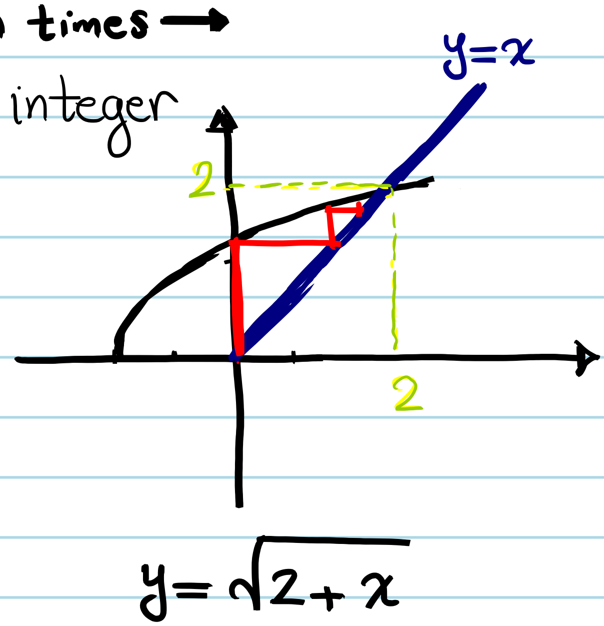
In the class, we showed that if $d \mid m$ and $d \mid n$, then $d \mid m+n$. Now notice that $d \mid \ell_i \Rightarrow d \mid s_i \ell_i$.]

5.(a) Define the following sequence of real numbers recursively

$$a_n = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ times}}.$$

(b) Prove that for any positive integer

$$a_n \le a_{n+1}.$$



[**Hint** ① Use induction on n;

② Show that

$$f(x) = \sqrt{2+x}$$

is increasing.]

(c) Prove that $a_n < 2$ for any positive integer n.

[**Hint**: ① Use induction on n;

② Show that $f(x) = \sqrt{2+x}$ is increasing and $f(2) = 2$.]

6. (Postage stamp problem) Prove that every postage greater than 34 can be obtained by stamps of denominations 5 and 9.

[**Hint**: ① Use strong induction on n, to show any postage $n \geq 34$ can be obtained by stamps of denominations 5 and 9.

② $34 = 5 \times 5 + 9$,

$35 = 7 \times 5$,

$36 = 4 \times 9$,

$37 = 2 \times 5 + 3 \times 9$,

$38 = 4 \times 5 + 2 \times 9$.]

[**Remark**: In this question, you prove that, for any integer $k \geq 34$, there are non-negative integers m and n such that

$$5n + 9m = k. \quad (*)$$

① If we allow m and n to be any integer (and not necessarily non-negative), then $(*)$ has a solution for any integer k.

② Compare this with problem 1. As it is discussed in class, the main difference is that 7 and 21 have a common divisor larger than 1 and 5 and 9 do not have a common divisor larger than 1.]