1 Propositional forms and truth-table.

1. Which one of the following propositional forms is NOT equivalent to \( P \Rightarrow Q \)? Justify your answer.
   (a) \((P \land (\neg Q)) \Rightarrow O\).
   (b) \((\neg Q) \Rightarrow (\neg P)\).
   (c) \((\neg (P \land Q))\).
   (d) \((\neg P) \lor Q\).

2. Show, without using truth tables, that the propositional form \((P \land Q) \lor (P \land \neg Q)\) is equivalent to \(P\).

3. Find a propositional form whose truth table is the following.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

2 Direct proof, case-by-case proof and proof by contradiction.

1. Prove that for all integers \(n\), \(4(n^2 + n + 1) - 3n^2\) is a perfect square.

2. Show that if a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.

3. Recall that a rotation matrix is a matrix of the form \[
\begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\] for some angle \(\theta\). Prove that \[
\begin{bmatrix}
1/2 & -1/2 \\
1/2 & 1/2
\end{bmatrix}
\] is not a rotation matrix. (You may use trig identities without proof.)

4. Prove that if \(d\) divides \(l_1\) and \(d\) divides \(l_2\), then \(d|s_1l_1 + s_2l_2\) for any integers \(s_1, s_2\).
5. Let $k$ be a positive integer. Prove that $2^k - 1$ and $2^k + 1$ have no common divisor larger than 1.

6. Prove that if $n^2$ is even, then $n$ is even. (Hint: An even number is one that can be written in the form $2k$ for some integer $k$.)

7. For a real number $x$, let $[x]$ be the integer part of $x$, i.e. it is the largest integer less than or equal to $x$.

   (a) Prove that $[-x] = -[x]$ if and only if $x$ is an integer. (You may use the fact that for a real number $x$ and an integer $n$ we have $[x] = n$ if and only if $n \leq x < n + 1$.)

   (b) Prove that $[x + m] = [x] + m$ for any real number $x$ and any integer $m$.

   (c) Prove that $[3x] = [x] + [x + \frac{1}{3}] + [x + \frac{2}{3}]$ for any real number $x$. (Hint: let $y = x - [x]$ and consider three cases separately (i) $0 \leq y < 1/3$, (ii) $1/3 \leq y < 2/3$, and (iii) $2/3 \leq y < 1$.)

3 Constructing a proof backwards and inequalities.

1. Prove that for any positive real numbers $x$ and $y$ we have

   $$\sqrt{xy} \leq \frac{x + y}{2}.$$ 

2. Let $a, b, c$ and $d$ be real numbers such that $a > b$ and $c > d$. Prove that

   $$ac + bd > ad + bc.$$ 

3. Let $x$ and $y$ be positive integers. Prove that

   $$\frac{x + y}{2} \leq \sqrt{\frac{x^2 + y^2}{2}}.$$ 

4. Let $a, b$ and $c$ be three real numbers. Prove that

   $$ab + ac + bc \leq a^2 + b^2 + c^2.$$ 

4 Proof by induction.

1. Give a recursive definition for each sequence:

   a) 1, 4, 7, 10, 13, 16, . . .

   b) -1, 1, -1, 1, -1, 1, . . .

   c) 1, 4, 9, 16, 25, 36, . . .

   d) 1, 4, 8, 13, 19, 26, . . .

   e) 8, 16, 32, 64, 128, 256, . . .

2. Show that $n^3 - n$ is divisible by 3 for every positive integer $n$.

3. Prove that for any positive integer $n$,

   a) $$\sum_{i=0}^{n} i = \frac{n(n + 1)}{2}$$
b) \[ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

c) \[ \sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4} \]

4. If the Fibonacci numbers are defined by \( f_0 = 0, f_1 = 1, \) and \( f_{n+1} = f_n + f_{n-1}, \) prove that for all \( n \geq 0, \)
\[ \sum_{i=0}^{n} f_i^2 = f_n f_{n+1}. \]

5. Let \( a_1 = 1 \) and \( a_{n+1} = \frac{3a_n + 1}{2a_n + 1} \) for any positive integer \( n. \) Prove that
   (a) For any positive integer \( n, \) we have that \( a_n < a_{n+1} \).
   (b) For any positive integer \( n, \) we have that \( a_n < \frac{1+\sqrt{3}}{2}. \)

6. Let \( a_1 = 1 \) and \( a_{n+1} = \sqrt{1 + a_n} \) for any positive integer \( n. \) Prove that
   (a) For any positive integer \( n, \) we have that \( a_n < a_{n+1} \).
   (b) For any positive integer \( n, \) we have that \( a_n < \frac{1+\sqrt{5}}{2}. \)

7. Let \( u_0 = 0, u_1 = 1 \) and \( u_{n+1} = 2u_n + 2u_{n-1}. \) Let \( A = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}. \) Prove that
   (a) For any positive integer \( n \) we have
   \[ A^n = \begin{bmatrix} 2u_{n-1} & u_n \\ 2u_n & u_{n+1} \end{bmatrix}. \]
   (b) For any positive integers \( m \) and \( n, \) we have that
   \[ u_{m+n} = 2u_{n-1}u_m + u_nu_{m+1}. \]
   (Hint: \( A^{m+n} = A^m A^n. \))
   (c) Use part (b) and induction to prove that \( u_m | u_{mk} \) for any positive integers \( m \) and \( k. \)