1. Determine the truth-value of the following propositions:

(a) There are no sets A and B such that

$$A \in B \quad \text{and} \quad A \subseteq B.$$

(b) $\{1,2\} \in \{1, \{1,2\}, \emptyset\} \wedge \{\emptyset\}$ has one element.

(c) If $a \in \{1,2,3\}$ and $\{1,2,3\} \cap \{a, 2\} = \{2\}$,

then $a = 1$.

(d) If $a \in \mathbb{Z}$ and $\{1,2,3\} \setminus \{a, 2\} = \{3\}$,

then $a = 1$.

2. One of the important axioms of set theory is the following:

(Axiom of regularity) For any non-empty set X, there is $Y \in X$ such that $X \cap Y = \emptyset$.

(a) Use the above axiom to prove that there is NO set A such that $A \in A$. (**Hint**. Consider $X = \{A\}$.)

(b) Use the above axiom to prove that there are NO sets A and B such that $A \in B$ and $B \in A$.

(**Hint**. Consider $X = \{A, B\}$)

3. Let X be a set and $a,b,c,d \in X$.

(a) Prove or disprove: $\{a,b\}=\{c,d\} \Rightarrow a=c \wedge b=d$.

(b) Prove or disprove: $\{a,\{a,b\}\}=\{c,\{c,d\}\}$

$$\Rightarrow a=c \wedge b=d.$$

(Hint. ① Use problem 2 to show $a \neq \{a,b\}$ and $c \neq \{c,d\}$.

② Use problem 2 to show $a \neq \{c,d\}$.

③ Show that $\{a,b\}=\{a,d\}$ implies $b=d$.)

4. Write down the negation of the following propositions:

(a) For every $\varepsilon \in \mathbb{R}^+$ there is $\delta \in \mathbb{R}^+$ such that

$$|x-1|<\delta \Rightarrow |x^2-1|<\varepsilon.$$

(b) For every $\varepsilon \in \mathbb{R}^+$ and $x \in \mathbb{R}$ there is $n \in \mathbb{Z}$ such that

$$|x-n|<\varepsilon.$$

(c) Let α be an irrational number, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For every $\varepsilon \in \mathbb{R}^+$ and $x \in \mathbb{R}$ there are $m,n \in \mathbb{Z}$ such that

$$|x-m-n\alpha|<\varepsilon.$$

(d) For any $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^{>0}$, there is a unique pair (q, r) of integers such that

$$a = bq + r \quad \text{and} \quad 0 \leq r < b.$$

5. (a) Prove or disprove: $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad y^2 > 2013 + x$.

(b) Prove or disprove: $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad y^3 > 2013 + x$.

(c) Prove or disprove:

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^{>0}, \quad n \geq N \Rightarrow \frac{1000}{n} < \varepsilon.$$

(**Hint**: In (a) and (b), we are seeking for a lower bound for the functions $y^2 - 2013$ and $y^3 - 2013$, respectively.

In (c), you are allowed to use the fact that

$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, \quad x < n$.)

6. (a) Prove that $\forall \delta > 0, \exists n \in \mathbb{Z}^{>0}, \quad \dfrac{1}{2n\pi} < \delta \ \wedge \ \dfrac{1}{(2n+1)\pi} < \delta$.

(b) Prove that there is NO $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0, \exists \delta > 0, \quad |x| < \delta \Rightarrow \left|\cos\left(\frac{1}{x}\right) - L\right| < \varepsilon.$$

(**Hint.** Suppose to the contrary that there is such $L \in \mathbb{R}$.
Then for $\varepsilon = 1/4$ there is $\delta > 0$ such that

$$|x| < \delta \implies \left|\cos\left(\frac{1}{x}\right) - L\right| < 1/4.$$

Use part (a) to find $x = \dfrac{1}{2n\pi}$ or $\dfrac{1}{(2n+1)\pi}$ such that
$|x| < \delta$ and conclude that

$$|1 - L| < 1/4 \quad \text{and} \quad |-1 - L| < 1/4.$$

Use $|a-b| \leq |a| + |b|$ to get a contradiction.)