1. Determine the truth-value of the following propositions:
   (a) There are no sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.
   (b) $\{1, 2, 3\} \in \{1, 2, 3, \emptyset\} \land \emptyset \in \emptyset$ has one element.
   (c) If $a \in \{1, 2, 3\}$ and $\{1, 2, 3\} \cap \{a, 2\} = \{2\}$, then $a = 1$.
   (d) If $a \in \mathbb{Z}$ and $\{1, 2, 3\} \setminus \{a, 2\} = \{3\}$, then $a = 1$.

2. One of the important axioms of set theory is the following:
   (Axiom of regularity) For any non-empty set $X$, there is $Y \in X$ such that $X \cap Y = \emptyset$.

(a) Use the above axiom to prove that there is NO set $A$ such that $A \in A$. (Hint: Consider $X = \{A\}$.)

(b) Use the above axiom to prove that there are NO sets $A$ and $B$ such that $A \in B$ and $B \in A$. (Hint: Consider $X = \{A, B\}$.)
3. Let $x$ be a set and $a, b, c, d \in x$.

(a) Prove or disprove: \( \exists a, b \exists c, d \Rightarrow a = c \land b = d \).

(b) Prove or disprove: \( \exists a, \exists a, b \exists c, \exists c, d \exists \Rightarrow a = c \land b = d \).

(Hint. 1) Use problem 2 to show $a \neq \exists a, b \exists$ and $c \neq \exists c, d \exists$.

2 Use problem 2 to show $a \neq \exists c, d \exists$.

3 Show that $\exists a, b \exists = \exists a, d \exists$ implies $b = d$.

4. Write down the negation of the following propositions:

(a) For every $\varepsilon \in \mathbb{R}^+$ there is $\delta \in \mathbb{R}^+$ such that \( |x - 1| < \delta \Rightarrow |x^2 - 1| < \varepsilon \).

(b) For every $\varepsilon \in \mathbb{R}^+$ and $x \in \mathbb{R}$ there is $n \in \mathbb{Z}$ such that \( |x - n| < \varepsilon \).

(c) Let $\alpha$ be an irrational number, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For every $\varepsilon \in \mathbb{R}^+$ and $x \in \mathbb{R}$ there are $m, n \in \mathbb{Z}$ such that \( |x - m - n\alpha| < \varepsilon \).
(d) For any $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^>$, there is a unique pair $(q, r)$ of integers such that

$$a = bq + r \quad \text{and} \quad 0 \leq r < b.$$ 

5. (a) Prove or disprove: $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad y^2 > 2013 + x$.

(b) Prove or disprove: $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad y^3 > 2013 + x$.

(c) Prove or disprove:

$$\forall \varepsilon \in \mathbb{R}, \exists N \in \mathbb{Z}^> : n > N \Rightarrow \frac{1000}{n} < \varepsilon.$$  

(Hint: In (a) and (b), we are seeking for a lower bound for the functions $y^2 - 2013$ and $y^3 - 2013$, respectively. In (c), you are allowed to use the fact that 

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, \ x < n.$$ )

6. (a) Prove that $\forall \delta > 0, \exists n \in \mathbb{Z}^> : \frac{1}{2n\pi} < \delta \wedge \frac{1}{(2n+1)\pi} < \delta$.

(b) Prove that there is NO $l \in \mathbb{R}$ such that

$$\forall \varepsilon > 0, \exists \delta > 0, \ |x| < \delta \Rightarrow |\cos\left(\frac{1}{x}\right) - 1| < \varepsilon.$$
(Hint. Suppose to the contrary that there is such \( I \in \mathbb{R} \).

Then for \( \varepsilon = 1/4 \) there is \( \delta > 0 \) such that

\[ |x| < \delta \Rightarrow |\cos(\frac{1}{x}) - I| < \frac{1}{4}. \]

Use part (a) to find \( x = \frac{1}{2n\pi} \) or \( \frac{1}{(2n+1)\pi} \) such that \( |x| < \delta \) and conclude that

\[ |1 - I| < \frac{1}{4} \quad \text{and} \quad |1 - I| < \frac{1}{4}. \]

Use \( |a - b| \leq |a| + |b| \) to get a contradiction.)