1. Let me recall from the lecture that a two-player game is called a finite game if it finishes (with a winner) in finitely many turns. We define length $\ell(G)$ of a game G to be the maximum number of turns needed to finish G.

In a finite Game, the winner is the person who does the last move (a player loses if s/he does not have a move to make.).

Let us define "sum" of two finite games G_1 and G_2.

G_1 and G_2 : 2-player, finite games

$G_1 \oplus G_2$: 2-player game.

On each turn, a player must make a move in one and only one of the games G_1 and G_2.

(a) Prove that $G_1 \oplus G_2$ is a finite game.

(Hint: Think about $\ell(G_1 \oplus G_2)$ in terms of $\ell(G_1)$ and $\ell(G_2)$.)

(b) Prove that for any finite 2-player game G, $G \oplus G$ is N (i.e. the second playe can win).

2. Prove that ① if G_1 is $\mathcal{P}$ (i.e. the first player can win) and G_2 is $\mathcal{N}$ (i.e. the second player can win), then

$G_1 \oplus G_2$ is $\mathcal{P}$. ② If G_1 and G_2 are $\mathcal{N}$, then

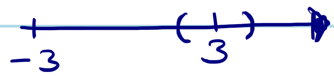
$G_1 \oplus G_2$ is $\mathcal{N}$.

(Hint: Prove ① and ② at the same time, by strong induction on $\ell(G_1)+\ell(G_2)$)

3. Prove that $\lim\limits_{x\to 3} x^2 = 9$.

(Hint: Go through the proof presented in the lecture and make the needed modifications!

- $x^2-9=(x-3)(x+3)$



and if $|x-3|\leq 1$, then $|x+3|\leq 7$.

- Show that $\delta = \begin{cases} 1 & \varepsilon \geq 7 \\ \varepsilon/7 & \varepsilon < 7 \end{cases}$ works.)

4. Prove that

$$\forall n \in \mathbb{Z}^{>1}, \left(\left(\nexists m \in \mathbb{Z}^{>1},\ m \leq \sqrt{n} \wedge m \mid n \right) \Rightarrow n \text{ is prime} \right)$$

(Hint. Prove it by contradiction.)

5. Let A, B, C and D be four sets.

(a) Prove or disprove:

$$(A\times B)\cup(C\times D)=(A\cup C)\times(B\cup D).$$

(b) Prove or disprove:

$$(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D).$$

(c) Prove or disprove:

$$(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D).$$

6. The following statements are either incomplete or false. Say why? And make the necessary modification.

(a) Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sqrt{x^2+1}$. Then $f=g_3\circ g_2\circ g_1$ where $g_1(x)=x^2$, $g_2(x)=x+1$ and $g_3(x)=\sqrt{x}$.

(Hint: domains and codomains.)

(b) Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x)=\frac{1}{x}$. Then

$$f\circ f=id._{\mathbb{R}},$$

where $id._{\mathbb{R}}:\mathbb{R}\to\mathbb{R}$, $id._{\mathbb{R}}(x)=x$.

7. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Prove that there is no $g: \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = x$ for any $x \in \mathbb{R}$.

(Hint ① Prove by contradiction.

② $1^2 = (-1)^2 = 1$.)

[Recall / Hint for the first two problems:

- A game is called a P-game if the first player has a winning strategy
- A game is called an N-game if the second player can win no matter how good the first player plays.
- A and B are playing a game G.

First player : A	after the first move $\longrightarrow$	First player : B
Second player : B		Second player : A
Game : G		Game : G′
		and $\ell(G') \leq \ell(G) - 1$.

- A game G is a P-game $\iff$

A has a move s.t. G changes to a game G′ which
existensial
is an N-game.

- A game G is an N-game $\iff$

  Any move of A changes G to a game which is

  universal

  a P-game.]