Let $R$ be a semisimple ring. Then

$$R = \bigoplus_{i=1}^{k} I_i,$$

where $I_i$ are minimal left $R$-modules. After relabeling $I_i$, we can and will assume that

$$R = \bigoplus_{i=1}^{l} \left( \bigoplus_{j=1}^{n_i} I_{i,j} \right),$$

such that $I_{i,j} \cong I_{i,j'}$ for any $1 \leq j \leq j' \leq n_i$ and $I_{i,j} \neq I_{i',j'}$ for any $i \neq i'$.

(By the way, this implies that $R \cong \bigoplus_{i=1}^{l} \left( I_{i,1}^{n_i} \right)$.)

@ Let $\phi \in \text{Hom}_R(I_{i,1}, R)$. Prove that

$$\text{Im}(\phi) \subseteq M_i := \bigoplus_{j=1}^{n_i} I_{i,j}.$$

(b) Prove that if $\phi \in \text{End}_R(R)$, then, for any $i$,

$$\phi(M_i) \subseteq M_i,$$

where $M_i := \bigoplus_{j=1}^{n_i} I_{i,j}$. (Hint: Use part @.)

© Prove $\text{End}_R(R) \cong \bigoplus_{i=1}^{l} \text{End}_R(M_i)$ as two rings.
6. Prove that $\text{End}_R(M_i) \cong M_{n_i}(D_i)$ where $D_i = \text{End}_R(I_{i,1})$ is a division ring.

7. Prove that $R \cong \bigoplus_{i=1}^{d} M_{n_i}(D_i)^{op}$.

(Hint: $\text{End}_R(R) \cong R^{op}$.)

(For a given ring $(A,+,\cdot)$, its opposite ring $(A^{op},+,\cdot)$ is a ring with the same underlying additive group and its multiplication is defined as follows:

$x \cdot y = y \cdot x$)

Exp 1. A commutative $\Rightarrow A = A^{op}$.

Exp 2. $A = M_n(F) \Rightarrow A \cong A^{op}$

$x \mapsto x^T$ where $x^T$ is the transpose of $x$.

Exp 3. $\tau: A \to A$ is called an involution if

1. $\tau^2 = \text{id}_A$.
2. $\tau(x+y) = \tau(x) + \tau(y)$.
3. $\tau(xy) = \tau(y) \cdot \tau(x)$.

If $A$ has an involution, then $A \cong A^{op}$.
Exp 4. If $D$ is a division ring, then $D^{op}$ is also a division ring.

Exp 5. $M_n(D)^{op} \sim M_n(D^{op})$ [Similar to Exp 2.]

[As a result of this exercise, you see that any (left) semisimple ring is isomorphic to

$$M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_k}(D_k)$$

where $D_i$ are division rings. One can prove that $n_i$ and $D_i$ are unique (up to isomorphism.) This is called Artin-Wedderburn theorem.]

2 [Expansion of Midterm Problem 3.]

Let $R$ be a commutative ring and

$$GL_n(R) = \{ A \in M_n(R) \mid \exists B \in M_n(R) : AB = BA = I_n \}.$$ 

Prove that $A \in GL_n(R)$ if and only if $\det(A)$ is a unit in $R$.

(You can find the following useful:

1. For any commutative ring $R$, one can define

$$\det : M_n(R) \rightarrow R$$
(2) \det(I_n) = 1 \text{ and } \det(AB) = \det(A) \det(B).

(3) The \((i,j)\) minor \(A_{ij}\) of \(A\) is the determinant of the \((n-1)\times(n-1)\) matrix that results from deleting the \(i^{th}\) row and the \(j^{th}\) column. The adjoint \(\text{adj}(A)\) of \(A\) is an \(n\times n\) matrix whose \((i,j)\) entry is

\((-1)^{i+j} A_{ji}\).

Then \(A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I_n\).

(b) Let \(R\) be a commutative ring and \(A, B \in M_n(R)\).

Prove that \(\text{Im}(A) \subseteq \text{Im}(B) \iff \exists X \in M_n(R): A = BX\).

\((\text{Im}(C) = \{ \sum_{i=1}^{n} R c_i^T \mid v^T \in R^n \})\)

(c) Let \(K\) be a field and \(R \subseteq K\) be a subring.

Assume \(A, B \in M_n(R)\) and \(\det(A) \neq 0\). Prove that

\(\text{Im}(A) = \text{Im}(B) \iff \exists P \in \text{GL}_n(R): A = BP\).

d) Let \(R\) be a commutative ring and \(P \in M_n(R)\). Prove that \(P \in \text{GL}_n(R)\) if and only if the columns of \(P\) form an \(R\)-basis of \(R^n\).
Let \( R \) be a commutative ring and \( A \in M_n(R). \)
Prove that \( \text{det}(A) R^n \subseteq \text{Im}(A). \)

Let \( K \) be a field and \( R \subseteq K \) be a PID.

Assume that \( A \in M_n(R) \) and \( \text{det}(A) \neq 0 \). Prove that

\[
\exists \vec{v}_1, \ldots, \vec{v}_n \in R^n \text{ and } q_1 | q_2 | \ldots | q_n \text{ s.t.}
\]

\[
(i) \quad R^n = R\vec{v}_1 \oplus \cdots \oplus R\vec{v}_n.
\]

\[
(ii) \quad \text{Im}(A) = Rq_1 \vec{v}_1 \oplus \cdots \oplus Rq_n \vec{v}_n.
\]

Conclude that \( \exists P_1 \in GL_n(R) \) s.t.

\[
\text{Im}(A) = \text{Im}(P_1 \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}).
\]

Now using (i), show that \( \exists P_1, P_2 \in GL_n(R) \) s.t.

\[
A = P_1 \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} P_2.
\]

Let \( F \) be a field. If \( A \in M_n(F[x]) \) and \( \text{det}(A) \neq 0 \),
then

\[
\dim_F \left( \frac{F[x]^n}{\text{Im}(A)} \right) = \deg(\text{det} A).
\]
3. Prove that the following are equivalent:
   a. \( R \) is a (left) semisimple ring.
   b. Any left \( R \)-module is semisimple.
   c. Any left \( R \)-module is projective.
   d. Any left \( R \)-module is injective.

   (You do not have to show that a and b are equivalent.)

4. Let \( R \) be a ring and \( M \) be an \( R \)-module. Prove that
   i. \( M \) is free.
   ii. \( M \) is projective.
   iii. \( M \) is flat.
   iv. \( M \) is torsion-free, i.e. \( ax = 0 \) if \( 0 \neq x \in M \) and \( a \in R \) is NOT a zero-divisor.

5. Prove that if \( R \) is a PID and \( M \) is a f.g. \( R \)-module, then i, ii, iii and iv are equivalent.
5. An abelian group $G$ is called divisible if for any $a \in G$ and any $n \in \mathbb{Z} \setminus \{0\}$, $n \chi = a$ has a solution in $G$.

(a) Prove that a $\mathbb{Z}$-module $G$ is injective if and only if $G$ is divisible.

(b) Let $G$ and $H$ be $\mathbb{Z}$-modules and $\phi \in \text{Hom}_{\mathbb{Z}}(G, H)$. If $G$ is divisible, then $\phi(G)$ is also divisible.

(c) If $G_i$ are divisible, then $\bigoplus_{i \in I} G_i$ and $\prod_{i \in I} G_i$ are also divisible.

(d) Prove that any abelian group can be embedded into a divisible abelian group.

(Hint: $\bigoplus_{i \in I} \mathbb{Z}/\mathbb{K} \to \bigoplus_{i \in I} \mathbb{Q}/\mathbb{K}$.)

(e) Let $J$ be a divisible abelian group. Prove that $\text{Hom}_{\mathbb{Z}}(R, J)$ is an injective $R$-module.

(f) Prove that any $R$-module can be embedded into an injective module.

(Hint: $\forall M: R$-mod $\exists J: \mathbb{Z}$-mod & divisible s.t.}
\[ M \xrightarrow{f} \mathcal{J} \text{ as } \mathbb{Z}-\text{modules} \Rightarrow \]

\[ M \simeq \text{Hom}_R(R,M) \xrightarrow{\cdot f} \text{Hom}_\mathbb{Z}(R,M) \xrightarrow{\cdot 1} \text{Hom}_\mathbb{Z}(R,\mathcal{J}) \cdot \]

6. Let \( P_1 \) and \( P_2 \) be projective \( R \)-modules. Prove that if

\[ 0 \to Q_1 \to P_1 \xrightarrow{\pi_1} M \to 0 \]

and

\[ 0 \to Q_2 \to P_2 \xrightarrow{\pi_2} M \to 0 \]

are short exact sequences, then

\[ P_1 \oplus Q_2 \simeq P_2 \oplus Q_1. \]

(Hint. Consider \( N = \{ (x_1, x_2) \in P_1 \oplus P_2 \mid \pi_1(x_1) = \pi_2(x_2) \} \))