1. [Infinite Galois Theory]

Let \( E/F \) be a normal and separable extension.

Let \( \Omega := \{ F \subseteq K \subseteq E \mid K/F \text{ finite, normal and separable} \} \)

@ Prove that \[ E = \bigcup_{K \in \Omega} K. \]

@ Let \( \phi: \text{Aut}(E/F) \rightarrow \prod_{K \in \Omega} \text{Gal}(K/F) \)

\[ \phi(\tau) := (\tau|_K)_{K \in \Omega} \]

Prove that \( \phi \) is a well-defined, injective group homomorphism.

@ Let \( \lim_{\Omega} \text{Gal}(K/F) := \{ (\tau_K)_{K \in \Omega} \mid \text{Gal}(K/F) \} \)

\[ \tau_K \Big|_{K'} = \tau_{K'}, \text{ if } K' \subseteq K \text{ and } K, K' \in \Omega. \]

Prove that \( \text{Im}(\phi) = \lim_{\Omega} \text{Gal}(K/F) \) and conclude that \( \text{Aut}(E/F) \cong \lim_{\Omega} \text{Gal}(K/F) \).

@ Take the discrete topology on the finite
sets $\text{Gal}(K/F)$ for any $K \in \Omega$ and prove that
with respect to the product topology $\prod_{K \in \Omega} \text{Gal}(K/F)$
is a closed subset of $\prod_{K \in \Omega} \text{Gal}(K/F)$.

© With respect to the above topology, prove that, for any $K \in \Omega$,

$$G_K := \{ \tau \in \text{Aut}(E/F) \mid \tau |_K = \text{id}_K \}$$
is an open subgroup of $\text{Aut}(E/F)$. And any neighborhood of the identity contains $G_K$
for some $K \in \Omega$.

[As it is discussed in class, it is called the
Krull topology.]

© Prove that, for any field $F \subseteq K \subseteq E$, $G_K$ is
a closed subgroup of $\text{Aut}(E/F)$.

© Prove that, for any subgroup $G$ of $\text{Aut}(E/F)$,

$$G_{\text{Fix}(G)} = \overline{G},$$
where $G_{\text{Fix}(G)}$ is defined as in © and $\overline{G}$ is the
closure of $G$ in $\text{Aut}(E/F)$.

[Hint: By (i), $\overline{G} \subseteq G_{\text{fix}(G)}$.]

Let $\mathcal{O}$ be an open subset of $\text{Aut}(E/F)$ which does not intersect $G$. By (ii), we can assume that

$\mathcal{O} = \mathcal{O}_o G_K$ for some $K \in \Omega$.

Let $\pi_K: \text{Aut}(E/F) \to \text{Gal}(K/F)$ be $\pi_K(\mathcal{O}) = \mathcal{O}_K$.

We have

1. $\pi_K(\mathcal{O}_o) \notin \pi_K(G)$

2. $G_{\text{fix}(G)} \cap K = G_{\text{fix}(\pi_K(G))}$.

3. $\{ \sigma \in \text{Gal}(K/F) \mid \sigma|_{G_{\text{fix}(\pi_K(G))}} = \text{id.} \}$

Hence $\mathcal{O}_o \notin G_{\text{fix}(G)}$.

(ii) Prove that there is a correspondence between closed subgroups of $\text{Aut}(E/F)$ and fields $F \subseteq K \subseteq E$.

$[G \mapsto G_{\text{fix}(G)}$ and $K \mapsto G_K]$. ]
2. [Separable and purely inseparable extensions]

Let \( F \) be a field of characteristic \( p > 0 \). Recall that an irreducible polynomial \( f(x) \in F[x] \) is separable if and only if \( f'(x) \neq 0 \).

(a) Prove that, if \( f(x) \in F[x] \) is irreducible, then

\[
 f(x) = f_{sep}(x^p)
\]

where \( f_{sep}(x) \in F[x] \) is an irreducible polynomial.

[Hint: (1) \( g'(x) = 0 \iff \exists h(x) \in F[x]: g(x) = h(x^p) \).

(2) \( h(x^p) \) irreducible \( \Rightarrow \) \( h(x) \) irreducible.]

(b) Let \( E/F \) be an algebraic extension. Prove that the following statements are equivalent:

(i) \( \forall \alpha \in E, \alpha^p \in F \) for some \( k \).

(ii) \( \forall \alpha \in E, m_{\alpha,F}(x) = x^p - \alpha \) for some \( \alpha \in F \).

(iii) \( \forall \alpha \in E/F, \alpha \) is NOT separable over \( F \).

[Hint: (ii) \( \Rightarrow m_{\alpha,F}(x) \mid x^p - \alpha^p = (x - \alpha)^p \)

\( \Rightarrow m_{\alpha,F}(x) = (x - \alpha)^n \), in particular \( \alpha^n \in F \).]
\[ \alpha \neq 0 \implies \text{Ord} \left( \alpha F^* \right) \mid p^k \implies \text{Ord} \left( \alpha F^* \right) = p^l \mid n. \]
\[ (x - \alpha) = x - \alpha \mid m_{\alpha, F}(x) \rightsquigarrow \bullet \]
\[ \text{[Use @.]} \]

[E/F is called a purely inseparable extension if the above properties hold.]

[Observe that E/F separable and purely inseparable imply that E=F.]

© Let E/F be an algebraic extension. Let’s recall that the separable closure

\[ E_{\text{sep}} := \{ \alpha \in E \mid m_{\alpha, F}(x) \text{ separable}\} \]

of F in E is a field. Prove that E/E_{\text{sep}} is purely inseparable.

© Prove that, if E/F and K/E are algebraic separable extensions, then K/F is an algebraic separable extension.

[Hint: E \subseteq K_{\text{sep}} \implies K/K_{\text{sep}} is separable and...
purely inseparable $\Rightarrow K = K_{sep}$.

F is called a perfect field if any algebraic extension $E/F$ is separable. Prove that F is perfect if and only if $F^p = F$.

[Hint: $(\Rightarrow)$ $aeF \setminus F^p \Rightarrow x^p - a \in F[x]$ is irreducible. $(\Leftarrow)$ If $g(x) = a_n x^n + \ldots + a_0 \in F[x]$, then $g(x^p) = h(x)^p$ where $h(x) = b_n x^n + \ldots + b_0$ and $b_i^p = a_i$.]

3. Let $\tau \in \text{Aut}(\overline{Q}/\mathbb{Q})$ and $F = \text{Fix}(\tau)$. Prove that, if $E/F$ is (finite and) Galois, then $\text{Gal}(E/F)$ is cyclic.

4. Let $F$ be a maximal subfield of $\overline{Q}$ which does not contain $\sqrt[3]{3}$. Prove that, if $E/F$ is (finite and) Galois, then $\text{Gal}(E/F)$ is cyclic.

[Hint: Let $G = G_F \subseteq G_{\overline{Q}} = \text{Aut}(\overline{Q}/\mathbb{Q})$, $N = G_{\mathbb{Q}[\sqrt{3}]}$, and $\pi : G_{\overline{Q}} \rightarrow G_{\overline{Q}}/N \cong \text{Gal}(\mathbb{Q}[\sqrt{3}]/\mathbb{Q})$.]
\[ \mathbb{Q}[\sqrt{3}] \not\subseteq F \implies G_F \not\subseteq N \implies |\pi(G_F)| = 2. \]

\[ (F \not\subseteq E \implies \mathbb{Q}[\sqrt{3}] \subseteq E) \implies \left( H \not\subseteq G_F \implies |\pi(H)| = 1 \right) \]

\[ \text{closed} \]

\[ \text{If } \pi(g) \neq 1, \text{ then } \langle g \rangle = G_F. \]