1. (5 points) Determine the number of conjugacy classes of $\text{GL}_2(\mathbb{F}_q)$. (Hint: use linear algebra)

2. (10 points) Prove that any projective module is flat.

3. (10 points) Let $E/F$ be a (finite) Galois extension. Prove that as $E$-algebras

$$E \otimes_F E \simeq E \oplus \cdots \oplus E,$$

where the right hand side is the sum of $[E:F]$-many terms.

4. Let $E/F$ be a normal finite extension and $\text{char}(F) = p > 0$.

   (a) (5 points) Prove that there is $f(x) \in F[x]$ such that $E$ is the splitting field of $f(x)$ over $F$.
   
   (b) (5 points) Let $F_s := \{ a \in E \mid a$ is separable over $F \}$. Prove that $F_s$ is a field and $F_s/F$ is Galois.
   
   (c) (10 points) Prove that the restriction map induces an isomorphism between $\text{Aut}(E/F)$ and $\text{Aut}(F_s/F)$.

5. (10 points) Let $f(x) \in F[x]$ be an irreducible and separable polynomial. Assume $\text{deg}(f) = p$ is prime and $G$ is the Galois group of $f$. Prove that $p || |G|$ and $p^2 \nmid |G|$.

6. (a) (10 points) Let $f(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree $d$. Prove that

$$d | n \text{ if and only if } f(x) | x^{p^n} - x.$$

(b) (5 points) Let $A_d := \{ f(x) \in \mathbb{F}_p[x] \mid f(x)$ is irreducible, monic and of degree $d \}$ and $a_d := |A_d|$. Prove that

$$\sum_{d|n} da_d = p^n.$$