1. Suppose to the contrary that there integers m and n such that

$$7n+21m=51.$$

Then $51=7(n+3m)$. So $2=7(n+3m)-49=7(n+3m-7)$.

Hence $7\mid 2$ which is a contradiction. (This contradicts the following fact that we proved in the class:

$$\left.\begin{array}{l} d\mid n \\ 0<d,n \end{array}\right\} \Rightarrow d\le n.\text{)}$$

■

2. We proceed by induction on n.

Base of the induction. $3^1=3\geq 2=1+1$.

The induction step. $3^k\geq k+1 \Rightarrow 3^{k+1}\geq (k+1)+1=k+2$ (?)

$$3^{k+1}=3\times 3^k \geq 3(k+1) \quad \text{(by the induction hypothesis.)}$$
$$=3k+3$$
$$\geq k+2. \quad (k\geq 0)$$

■

3.(a) As it was discussed in the class, we proceed by induction on n.

Base of the induction. $A=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}f_0 & f_1\\ f_1 & f_2\end{bmatrix}$ ✓

The induction step. $A^k=\begin{bmatrix}f_{k-1} & f_k\\ f_k & f_{k+1}\end{bmatrix} \Longrightarrow A^{k+1}=\begin{bmatrix}f_k & f_{k+1}\\ f_{k+1} & f_{k+2}\end{bmatrix}$ (?)

$$A^{k+1}=A^k\cdot A=\underbrace{\begin{bmatrix}f_{k-1} & f_k\\ f_k & f_{k+1}\end{bmatrix}}_{\text{(by the induction step)}}\cdot\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}f_k & f_{k-1}+f_k\\ f_{k+1} & f_k+f_{k+1}\end{bmatrix}=\begin{bmatrix}f_k & f_{k+1}\\ f_{k+1} & f_{k+2}\end{bmatrix}.$$

■

(b) $A^{m+n} = \begin{bmatrix} f_{m+n-1} & f_{m+n} \\ f_{m+n} & f_{m+n+1} \end{bmatrix}$ (by part (a)).

On the other hand, $A^{m+n} = A^m \cdot A^n = \begin{bmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{bmatrix} \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$

$$= \begin{bmatrix} f_{m-1} f_{n-1} + f_m f_n & f_{m-1} f_n + f_m f_{n+1} \\ f_m f_{n-1} + f_{m+1} f_n & f_m f_n + f_{m+1} f_{n+1} \end{bmatrix}$$

Comparing the $\underline{1,2}$ entries, we have

$$\mathbf{f_{m+n} = f_{m-1} f_n + f_m f_{n+1}}.$$

4. We prove by induction on k that $f_m \mid f_{mk}$ for any positive integers m and k.

Base of the induction: $f_m \mid f_m$ ✓.

The induction step: $f_m \mid f_{mk} \Longrightarrow f_m \mid f_{m(k+1)}$ (?)

$f_{m(k+1)} = f_{m+mk} = f_{m-1} f_{mk} + f_m f_{mk+1}$ (by the previous problem.)

$$\left.\begin{array}{l} f_m \mid f_m \\ f_m \mid f_{mk} \text{ (by the induction hypothesis)} \end{array}\right\} \Longrightarrow f_m \mid f_{m-1} f_{mk} + f_{mk+1} f_m = f_{m(k+1)}.$$

(Here we are using the following fact given as Hint 3:

$$\left.\begin{array}{l} d \mid \ell_1 \\ d \mid \ell_2 \end{array}\right\} \Longrightarrow d \mid s_1 \ell_1 + s_2 \ell_2$$ for any integers s_1 and s_2.) ∎

5.(a) $a_{n+1} = \sqrt{2 + a_n}$ and $a_1 = \sqrt{2}$.

(b) **Claim.** $f(x)=\sqrt{2+x}$ is increasing. (We assume $x\geq 0$)

Pf of the claim $x>y\geq 0 \Rightarrow x+2>y+2>0$

$\Rightarrow f(x)>f(y)>0.$ ∎

We proceed by induction on n.

Base of the induction. $\sqrt{2}\leq\sqrt{2+\sqrt{2}} \Leftarrow 2\leq 2+\sqrt{2} \Leftarrow 0\leq\sqrt{2}.$

The induction step. $a_k\leq a_{k+1} \Rightarrow a_{k+1}\leq a_{k+2}$ (?)

$$a_k\leq a_{k+1} \underset{\text{(by the above claim)}}{\Rightarrow} f(a_k)\leq f(a_{k+1}) \Rightarrow a_{k+1}\leq a_{k+2}.$$

(c) Again we proceed by induction on n.

Base of induction. $\sqrt{2}<2 \Leftarrow 1<\sqrt{2} \Leftarrow 1<2.$

The induction step. $a_k<2 \Rightarrow a_{k+1}<2$ (?)

$$a_k<2 \underset{\text{(by the above claim)}}{\Rightarrow} f(a_k)<f(2) \Rightarrow a_{k+1}<2.$$ ∎

6. We proceed by strong induction.

Base of the induction. $34=5\times 5+9$. (This means 5 stamps of 5 cents and 1 stamp of 9 cents.)

The strong induction step.

We assume for any integer $34\leq k\leq n$ a postage of k cents can be obtained using only stamps of denominations 5 and 9.

We would like to prove that a postage of $n+1$ cents can be obtained using stamps of 5 cents and 9 cents.

As it is given in the hint, we can do this if $34\leq n+1\leq 38$:

$35 = 5 \times 7$, $36 = 9 \times 4$, $37 = 5 \times 2 + 9 \times 3$, $38 = 5 \times 4 + 9 \times 2$.

So we can assume $39 \leq n+1$. Thus $34 \leq (n+1) - 5 \leq n$.

Hence by the strong induction hypothesis a postage of $(n+1)-5$ cents can be obtained using only stamps of denominations 5 and 9. So a postage of $n+1 = [(n+1)-5] + 5$ cents can also be obtained using only stamps of denominations 5 and 9. $\blacksquare$