1. Suppose to the contrary that there integers \( m \) and \( n \) such that

\[
7n + 21m = 51.
\]

Then \( 51 = 7(n+3m) \). So \( 2 = 7(n+3m) - 49 = 7(n+3m) - 7 \cdot 7 \cdot 7 \).

Hence \( n \), which is a contradiction. (This contradicts the following fact that we proved in the class:

\[
d \mid n \implies d \leq n.
\]

\( 0 < d, n \)

2. We proceed by induction on \( n \).

**Base of the induction.** \( 3^1 = 3 \geq 2 = 1+1 \).

**The induction step.** \( 3^k \geq k+1 \) \( \implies 3^{k+1} \geq (k+1)+1=k+2 \) \( (2) \)

\[
3^{k+1} = 3 \times 3^k \geq 3(k+1) \quad \text{(by the induction hypothesis)}
\]

\[
= 3k+3 \geq k+2. \quad (k \geq 0)
\]

3. (a) As it was discussed in the class, we proceed by induction on \( n \).

**Base of the induction.** \( A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_0 & f_1 \\ f_1 & f_2 \end{bmatrix} \ check \)

**The induction step.** \( A^k = \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix} \implies A^{k+1} = \begin{bmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{bmatrix} \) \( (2) \)

\[
A^{k+1} = A \cdot A = \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_k & f_{k-1}+f_k \\ f_{k+1} & f_{k+1}+f_{k+2} \end{bmatrix} = \begin{bmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{bmatrix}
\]

(by the induction step)
(b) \( A^{m+n} = \begin{bmatrix} f_{m+n-1} & f_{m+n} \\ f_{m+n} & f_{m+n+1} \end{bmatrix} \) (by part (a)).

On the other hand, \( A^n A^m = \begin{bmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{bmatrix} \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix} = \begin{bmatrix} f_{m-1} f_{n-1} + f_m f_n & f_{m-1} f_n + f_m f_{n+1} \\ f_{m-1} f_n + f_m f_{n+1} & f_{m-1} f_{n+1} + f_m f_{n+2} \end{bmatrix} \)

Comparing the 1,2 entries, we have

\[ f_{m+n} = f_{m-1} f_n + f_m f_{n+1}. \]

4. We prove by induction on \( k \) that \( f_m \mid f_{mk} \) for any positive integers \( m \) and \( k \).

**Base of the induction:** \( f_m \mid f_m \) \( \checkmark \).

**The induction step:** \( f_m \mid f_{mk} \implies f_m \mid f_{m(k+1)} \) (\( \ast \))

\[ f_{m(k+1)} = f_{m+mk} = f_{m-1} f_{mk} + f_m f_{mk+1} \] (by the previous problem.)

\[ f_m \mid f_m \]

\[ f_m \mid f_{mk} \] (by the induction) \( \begin{cases} \iff f_m \mid f_{m-1} f_{mk} + f_{mk+1} f_m = f_{m(k+1)} \\ \text{(by the induction)} \end{cases} \)

(Here we are using the following fact given as Hint 3:

\[ d \mid l_1, l_2 \implies d \mid s_1 l_1 + s_2 l_2 \] for any integers \( s_1 \) and \( s_2 \).)

5. (a) \( a_{n+1} = \sqrt{2 + a_n} \) and \( a_1 = \sqrt{2} \).
(b) \textbf{Claim.} \( f(x) = \sqrt{2 + x} \) is increasing. (We assume \( x \geq 0 \))

\textit{Proof of the claim} \( x > y \geq 0 \Rightarrow x + 2 > y + 2 > 0 \)
\( \Rightarrow f(x) > f(y) > 0 \).

We proceed by induction on \( n \).

\textit{Base of the induction.} \( \sqrt{2} \leq \sqrt{2 + \sqrt{2}} \iff 2 \leq 2 + \sqrt{2} \iff 0 \leq \sqrt{2} \).

\textit{The induction step.} \( a_k \leq a_{k+1} \implies a_{k+1} \leq a_{k+2} \) (\(?\))

\[ a_k \leq a_{k+1} \quad \implies \quad f(a_k) \leq f(a_{k+1}) \quad \implies \quad a_{k+1} \leq a_{k+2} . \]
(by the above claim)

(c) Again we proceed by induction on \( n \).

\textit{Base of induction.} \( \sqrt{2} < 2 \iff 1 < \sqrt{2} \iff 1 < 2 \).

\textit{The induction step.} \( a_k < 2 \implies a_{k+1} < 2 \) (\(?\))

\[ a_k < 2 \quad \implies \quad f(a_k) < f(2) \quad \implies \quad a_{k+1} < 2 . \]
(by the above claim)

6. We proceed by strong induction.

\textit{Base of the induction.} \( 34 = 5 \times 5 + 9 \). (This means 5 stamps of 5 cents and 1 stamp of 9 cents.)

\textit{The strong induction step.}

We assume for any integer \( 34 \leq k \leq n \) a postage of \( k \) cents can be obtained using only stamps of denominations 5 and 9.

We would like to prove that a postage of \( n + 1 \) cents can be obtained using stamps of 5 cents and 9 cents.

As it is given in the hint, we can do this if \( 34 \leq n + 1 \leq 38 \):
$35 = 5 \times 7$, $36 = 9 \times 4$, $37 = 5 \times 2 + 9 \times 3$, $38 = 5 \times 4 + 9 \times 2$.

So we can assume $39 \leq n+1$. Thus $34 \leq (n+1) - 5 \leq n$.

Hence by the strong induction hypothesis a postage of $(n+1) - 5$ cents can be obtained using only stamps of denominations 5 and 9. So a postage of $n+1 = [(n+1) - 5] + 5$ cents can also be obtained using only stamps of denominations 5 and 9.