1. (The First Solution)

We proceed by induction on $n$.

**Base of the induction:** $8^1 + 6 = 14 = 2 \times 7 \Rightarrow 7 \mid 8^1 + 6$.

**The induction step:** $7 \mid 8^k + 6 \Rightarrow 7 \mid 8^{k+1} + 6$ (2)

$8^{k+1} + 6 = 8[(8^k - 6) + 6]$ (We write it this way to use the induction hypothesis)

$= 8[7 \cdot 7 - 6] + 6$ (by the induction hypothesis $7 \mid 8^k + 6$)

$= 56 \cdot 7 - 42$

$= 7(8^k - 6)$

$\Rightarrow 7 \mid 8^{k+1} + 6$.

By induction on $n$, we prove that $7 \mid 6^n - (-1)^n$.

**Base of the induction:** $6^1 - (-1)^1 = 7 \Rightarrow 7 \mid 6^1 - (-1)^1$.

**The induction step:** $7 \mid 6^k - (-1)^k \Rightarrow 7 \mid 6^{k+1} - (-1)^{k+1}$ (2)

$6^{k+1} - (-1)^{k+1} = 6[(6^k - (-1)^k) + (-1)^k] - (-1)^{k+1}$

$= 6[7 \cdot 7 + (-1)^k] + (-1)^k$ [by the induction hypothesis]

$= 7[6^k + (-1)^k]$

$\Rightarrow 7 \mid 6^{k+1} - (-1)^{k+1}$.

If $n$ is even, then $7 \mid 6^n - 1 \Rightarrow 7 \mid (6^n - 1) + 7 = 6^n + 6$.

If $n$ is odd, then $7 \mid 6^n + 1$. If to the contrary $7 \mid 6^n + 6$, then

$7 \mid (6^n + 6) - (6^n + 1) = 5$

which is a contradiction.

Hence $7 \mid 6^n + 6 \iff n$ is even.

(The second solution.) In the last week's homework assignment, you proved that

$a \mid b_1 - c_1 \Rightarrow a \mid b_1 b_2 - c_1 c_2$. 

$a \mid b_2 - c_2$
Using this result, by induction on \( n \), we prove that
\[
 a \mid b - c \implies a \mid b^n - c^n.
\]

**Base of the induction.** \( a \mid b^1 - c^1 \). ✓

**The induction step.** \( a \mid b^k - c^k \implies a \mid b^{k+1} - c^{k+1} \) \((\star)\)

\[
 a \mid b^k - c^k \implies a \mid b^{k+1} - c^{k+1}.
\]

\( a \mid b - c \) \( \square \) (The mentioned problem.)

\[
\begin{align*}
\Rightarrow & 7 \mid 8 - 1 \implies 7 \mid 8^n - 1, \\
\Rightarrow & 7 \mid 6^n - 1.
\end{align*}
\]

(*The third solution*) If \( a \mid b - c \), then for any positive integer \( n \) we have

\[
 b^n - c^n = (b-c)(b^{n-1} + b^{n-2}c + b^{n-3}c^2 + \cdots + bc^{n-2} + c^{n-1})
\]

\[
= a \cdot 7 \left( \sum_{i=0}^{n-1} b^i c^{n-1-i} \right) \implies a \mid b^n - c^n.
\]

Now we can continue as above. \( \square \)

2. Let \( f(x) = \frac{6x+5}{x+2} = 6 - \frac{7}{x+2} \).

**Claim 1.** \( f(x) > 0 \) if \( x > 0 \).

**Pf of claim 1.** \( x > 0 \implies x+2 > 2 \) and \( 6x+5 > 5 \)

\[
\Rightarrow f(x) > 0.
\]

**Claim 2.** \( f(x) \) is increasing if \( x > 0 \).

**Pf of claim 2.** \( x > y > 0 \implies x+2 > y+2 > 0 \)

\[
\begin{align*}
\Rightarrow & \frac{1}{y+2} > \frac{1}{x+2}, \\
\Rightarrow & \frac{-7}{x+2} > \frac{-7}{y+2}, \\
\Rightarrow & f(x) > f(y).
\end{align*}
\]

By induction on \( n \), we prove that \( 0 < a_n < 5 \).
Base of induction. \(a_1 = 1 < 5\).

The induction step. \(a_k < 5 \implies a_{k+1} < 5\) (2)

We know that \(a_{k+1} = f(a_k)

\(a_k < 5\) (by the induction hypothesis) \(\implies\) \(f(a_k) = a_{k+1}\)

(by Claim 1.)

\(a_k < 5\) (by the induction hypothesis) \(\implies\) \(f(a_k) < f(5)\)

(by Claim 2.)

\(\implies a_{k+1} < 5\).

3. We proceed by induction on \(n\).

Base of the induction. \(1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} = \frac{2+1}{2 \times 2} \sqrt{\frac{3}{4}} = \frac{2+1}{2 \times 2}\)

The induction step

\[
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) \left(1 - \frac{1}{(k+1)^2}\right)
\]

\[
= \frac{k+1}{2k} \left(\frac{(k+1)^2 - 1}{(k+1)^2}\right)
\]

(by the induction hypothesis)

\[
= \frac{k+1}{2k} \cdot \frac{(k+1)(k+2)}{(k+1)^2}
\]

\[
= \frac{(k+1) \cdot k \cdot (k+2)}{2k \cdot (k+1)^2}
\]

\[
= \frac{k+2}{2(k+1)}
\]

4. As it was discussed in class, we proceed by strong induction.

Base of the induction. \(n = 2 \sqrt{\frac{3}{4}} \) (as 2 is a prime.)
The strong induction step: We assume any integer \(2 \leq k \leq n\) can be written as a product of primes. We would like to prove that \(n+1\) can be written as a product of primes.

If \(n+1\) is a prime, there is nothing to prove.

If \(n+1\) is not a prime, then there is an integer \(d\) such that

1. \(d \mid n+1\),
2. \(1 < d < n+1\).

Hence \(n+1 = d \cdot \left( \frac{n+1}{d} \right)\) and \(2 \leq d, \frac{n+1}{d} \leq n\). So by the strong induction hypothesis, \(d\) and \(\frac{n+1}{d}\) can be written as a product of primes.

Therefore \(n+1 = d \cdot \left( \frac{n+1}{d} \right)\) can be written as a product of primes. ■

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<thead>
<tr>
<th>(x \in A)</th>
<th>(x \in B)</th>
<th>(x \in A \setminus B)</th>
<th>(x \in B \setminus A)</th>
<th>(x \in A \Delta B)</th>
<th>(x \in A \cup B)</th>
<th>(x \in A \cap B)</th>
<th>(x \in (A \cup B) \setminus (A \cap B))</th>
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So \(x \in A \Delta B \equiv x \in (A \cup B) \setminus (A \cap B)\). Hence \(A \Delta B = (A \cup B) \setminus (A \cap B)\).

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<tr>
<th>(x \in A)</th>
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So, \( x \in A \triangle (B \cap C) \equiv x \in (A \cup B) \triangle C \). Thus, \( A \triangle (B \cap C) = (A \cup B) \triangle C \).

To write the above table, we used 5.a’s truth-table; namely we know:

- \( x \in S \triangle T \) is true if and only if one and exactly one of the following is true: 1) \( x \in S \) \text{ or } 2) \( x \in T \).

6. \( A \triangle C = B \implies A \triangle (A \triangle C) = A \triangle B \)

\[ \implies (A \triangle A) \triangle C = A \triangle B \]  \hspace{1cm} \text{(By 5.b)}

\[ \implies \emptyset \triangle C = A \triangle B \]  \hspace{1cm} \text{(By 5.c)}

\[ \implies C = A \triangle B \]  \hspace{1cm} \text{(By 5.d)}

So if there is such subset, then it is unique and it has to be \( A \triangle B \).

Thus shows that \( A \triangle B \) is such subset.