1. In (a) and (b), Domain = The set of the first component values. It is graph of a function if and only if for a given value x in the domain one and only one pair of the form $(x, \cdot)$ appears in the set.

(a) It is graph of a function. Its domain $= \{1, 2, 3\}$.

Its image $= \{1, 4\}$ (The set of the second component values.) For instance it is graph of

$f: \{1,2,3\} \to \{1,4\}$, $f(1) = 1$, $f(2) = 1$ and $f(3) = 4$.

(b) It is NOT graph of a function as it is NOT well-defined at 1.

(c) It is NOT graph of a function as it is NOT well-defined at 1.

(d) It is NOT graph of a function $f: \{1,2,3\} \to \{x, y, z, t\}$ as it is NOT defined at 2.

(e) It is graph of the function $f: \{1,2,3\} \to \{x, y, z, t\}$ such that $f(1) = x$, $f(2) = z$, $f(3) = y$

Hence domain $= \{1,2,3\}$ and $\text{Im}(f) = \{x, z, y\}$.

2. We have to show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$$f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$
$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$
$$\Rightarrow I_A(x_1) = I_A(x_2)$$
$$\Rightarrow x_1 = x_2.$$

3. We have to show $\forall b \in B, \exists a \in A, \; f(a) = b$. (I)

It is enough to notice that

$$b = I_B(b) = (f \circ g)(b) = f(g(b)),$$

Hence $a = g(b) \in A$ satisfies (I).

Alternative. f is onto $\Longleftrightarrow \forall b \in B, \; f^{-1}(b) \neq \emptyset$.

$$f(g(b)) = (f \circ g)(b) = I_B(b) = b \Rightarrow g(b) \in f^{-1}(b)$$
$$\Rightarrow f^{-1}(b) \neq \emptyset \Rightarrow f \text{ is onto}.$$

4. (Bonus Problem)

Converse of problem 2. If f is injective, then there is $g: B \to A$ s.t. $g \circ f = I_A$.

Pf. Let $a_0 \in A$ be a fixed (arbitrary) element of A.

Define $g: B \to A$ as follows

$$g(b) = \begin{cases} a_0 & b \notin \text{Im}(f) \\ a & b \in \text{Im}(f) \text{ where } a \text{ is the unique element of } A \text{ s.t. } f(a) = b. \end{cases}$$

Notice that, if $b \in \text{Im}(f)$, then $\exists a \in A$, $f(a) = b$ and it is unique as f is injective.

Claim. $g \circ f = I_A$

Pf. $(g \circ f)(a) = g(f(a)) = a$

(By the definition of g and the fact that $b = f(a) \in \text{Im}(f)$.) ∎

Converse of problem 3 If f is surjective, then there is $g: B \to A$ s.t. $f \circ g = I_B$.

Pf. $\forall b \in B$, $f^{-1}(b) \neq \emptyset$ (since f is onto.) $\Rightarrow \forall b \in B$, $\exists a_b \in f^{-1}(b)$. (choose one element)

$\Rightarrow$ Let $g: B \to A$, $g(b) := a_b$.

Claim $f \circ g = I_B$

Pf. $(f \circ g)(b) = f(g(b)) = f(a_b) = b$. ∎

Remark. In the above argument, we are using an axiom of set theory which is called the axiom of choice. One version of this axiom states:

Let I be a set and $\{A_i\}_{i\in I}$ be a family of non-empty sets. Then there is a function

$$f: I \to \bigcup_{i\in I} A_i \quad (\text{union of } A_i\text{'s})$$

s.t. $f(i) \in A_i$ for any $i \in I$.

5. Yes, there is such function $f: X \to X$. Let $a_0 \in A$ be a fixed (arbitrary) element of A. Let

$$f(x) = \begin{cases} x & x \in A \\ a_0 & x \notin A \end{cases}$$

Claim $\operatorname{Im}(f) = A$.

Pf. $y \in \operatorname{Im}(f) \Rightarrow y = f(x)$ for some $x \in X$.

$$\left.\begin{aligned} x\in A &\Rightarrow y = f(x) = x \in A \\ x \notin A &\Rightarrow y = f(x) = a_0 \in A \end{aligned}\right\} \Rightarrow y \in A$$

This implies $\operatorname{Im}(f) \subseteq A$ (I)

$a \in A \Rightarrow f(a) = a \Rightarrow a \in \operatorname{Im}(f)$

This implies $A \subseteq \mathrm{Im}(f)$ (II)

(I), (II) $\Rightarrow$ $A = \mathrm{Im}(f)$.

6. (a) We have to show $(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow a_1 = a_2$.

$$(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$$

$$\underset{\text{injective}}{\overset{g \text{ is}}{\Longrightarrow}} \quad f(a_1) = f(a_2)$$

$$\underset{\text{injective}}{\overset{f \text{ is}}{\Longrightarrow}} \quad a_1 = a_2 .$$

(b) We have to show $\forall c \in C, \exists a \in A, (g \circ f)(a) = c$.

Since g is onto, $\forall c \in C, \exists b_c \in B$ s.t. $g(b_c) = c$

Since f is onto, $\exists a_{b_c} \in A$ s.t. $f(a_{b_c}) = b_c$.

So $(g \circ f)(a_{b_c}) = g(f(a_{b_c}))$

$= g(b_c)$

$= c$.

7. Yes, for instance consider

$$f : (0,1) \longrightarrow \mathbb{R}, \quad f(x) = \mathrm{Cotan}(\pi x)$$

<u>1-1</u>. $f(x_1) = f(x_2) \Rightarrow \mathrm{cotan}(\pi x_1) = \mathrm{cotan}(\pi x_2)$

$\Rightarrow \pi x_1 = \pi x_2 + k\pi$ for some integer k.

$\Rightarrow \quad x_1 = x_2 + k$ for some integer k.

$$\left.\begin{array}{lll} 0 < x_1 < 1 & \Longrightarrow & \lfloor x_1 \rfloor = 0 \\ 0 < x_2 < 1 & \Longrightarrow & k < x_2 + k < k+1 \\ & \Longrightarrow & \lfloor x_2 + k \rfloor = k \\ & & x_1 = x_2 + k \end{array}\right\} \Rightarrow k = 0 \Rightarrow x_1 = x_2 .$$

Onto. From calculus, you know that

$$\left.\begin{array}{ll} (1) & f \text{ is continuous.} \\ (2) & \lim\limits_{x \to 1^-} f(x) = -\infty \\ (3) & \lim\limits_{x \to 0^+} f(x) = +\infty \end{array}\right\} \Rightarrow \text{by the mean value theorem we have that } \forall c \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = c$$

It is OK if a student refers to the graph of this function to show surjectivity: