1. In (a) and (b), Domain = The set of the first component values. It is graph of a function if and only if for a given value $x$ in the domain one and only one pair of the form $(x, \cdot)$ appears in the set.

(a) It is graph of a function. Its domain = $\{1, 2, 3\}$. Its image = $\{1, 4\}$ (The set of the second component values.) For instance it is graph of $\varphi: \{1, 2, 3\} \rightarrow \{1, 4\}$, $\varphi(1) = 1$, $\varphi(2) = 1$ and $\varphi(3) = 4$.

(b) It is NOT graph of a function as it is NOT well-defined at 1.

(c) It is NOT graph of a function as it is NOT well-defined at 1.

(d) It is NOT graph of a function $\varphi: \{1, 2, 3\} \rightarrow \mathbb{R}^{x,y,z,t}$ as it is NOT defined at 2.

(e) It is graph of the function $\varphi: \{1, 2, 3\} \rightarrow \mathbb{R}^{x,y,z,t}$ such that $\varphi(1) = x$, $\varphi(2) = z$, $\varphi(3) = y$. 
Hence \( \text{domain} = \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \) and \( \text{Im}(f) = \mathcal{X}_x, \mathcal{Z}, \mathcal{Y}_x \).

2. We have to show that \( f(x_1) = f(x_2) \implies x_1 = x_2 \).

\[
\begin{align*}
f(x_1) = f(x_2) \implies g(f(x_1)) &= g(f(x_2)) \\
\implies (g \circ f)(x_1) &= (g \circ f)(x_2) \\
\implies I_A(x_1) &= I_A(x_2) \\
\implies x_1 &= x_2.
\end{align*}
\]

3. We have to show \( \forall b \in B, \exists a \in A, f(a) = b \). (I)

It is enough to notice that

\[
b = I_B(b) = (f \circ g)(b) = f(g(b)),
\]

Hence \( a = g(b) \in A \) satisfies (I).

Alternative. \( f \) is onto \( \iff \forall b \in B, f^{-1}(b) \neq \emptyset \).

\[
f(g(b)) = (f \circ g)(b) = I_B(b) = b \implies g(b) \in f^{-1}(b) \\
\implies f^{-1}(b) \neq \emptyset \implies f \) is onto.
\]

4. (Bonus Problem)
Converse of problem 2. If \( f \) is injective, then there is \( g : B \to A \) s.t. \( g \circ f = I_A \).

Proof. Let \( a_0 \in A \) be a fixed (arbitrary) element of \( A \).
Define \( g: B \rightarrow A \) as follows:

\[
g(b) = \begin{cases} 
a_0 & \text{if } b \notin \text{Im}(f) \\
a & \text{if } b \in \text{Im}(f) \text{ where } \\
 & \text{\( a \) is the unique element of } A \\
 & \text{s.t. } f(a) = b.
\end{cases}
\]

Notice that, if \( b \in \text{Im}(f) \), then \( \exists a \in A, f(a) = b \) and it is unique as \( f \) is injective.

**Claim.** \( g \circ f = I_A \)

**Proof.** \( (g \circ f)(a) = g(f(a)) = a \)

(By the definition of \( g \) and the fact that \( b = f(a) \in \text{Im}(f) \).

Converse of Problem 3. If \( f \) is surjective, then there is \( g: B \rightarrow A \) s.t. \( f \circ g = I_B \).

**Proof.** \( \forall b \in B, f^{-1}(b) \neq \emptyset \implies \forall b \in B, \exists a_b \in f^{-1}(b). \) (Since \( f \) is onto.)

\[ \implies \text{Let } g: B \rightarrow A, g(b) := a_b. \]

**Claim** \( f \circ g = I_B \)

**Proof.** \( (f \circ g)(b) = f(g(b)) = f(a_b) = b. \)
Remark. In the above argument, we are using an axiom of set theory which is called the axiom of choice. One version of this axiom states:

Let $I$ be a set and $\{A_i\}$ be a family of non-empty sets. Then there is a function

$$f : I \to \bigcup_{i \in I} A_i$$

s.t. $f(i) \in A_i$ for any $i \in I$.

5. Yes, there is such function $f : X \to X$. Let $a_0 \in A$ be a fixed (arbitrary) element of $A$. Let

$$f(x) = \begin{cases} x & x \in A \\ a_0 & x \notin A \end{cases}$$

Claim. $\text{Im}(f) = A$.

Pf. $y \in \text{Im}(f) \implies y = f(x)$ for some $x \in X$.

$$x \in A \implies y = f(x) = x \in A \implies y \in A$$

$$x \notin A \implies y = f(x) = a_0 \in A$$

This implies $\text{Im}(f) \subseteq A$ (I)

$a \in A \implies f(a) = a \implies a \in \text{Im}(f)$
This implies $A \subseteq \text{Im}(f)$ (II)

(I), (II) $\implies A = \text{Im}(f)$.

6. (a) We have to show $(g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2$.

$(g \circ f)(a_1) = (g \circ f)(a_2) \implies g(f(a_1)) = g(f(a_2))$

$g$ is injective

$f(a_1) = f(a_2)$

$f$ is injective

$a_1 = a_2$.

(b) We have to show $\forall c \in C, \exists a \in A$, $(g \circ f)(a) = c$.

Since $g$ is onto, $\forall c \in C, \exists b_c \in B$ s.t. $g(b_c) = c$.

Since $f$ is onto, $\exists a_{bc} \in A$ s.t. $f(a_{bc}) = b_c$.

So $(g \circ f)(a_{bc}) = g(f(a_{bc})) = g(b_c) = c$.

7. Yes, for instance consider

$f: (0, 1) \to \mathbb{R}, \quad f(x) = \cotan(\pi x)$

1-1 $f(x_1) = f(x_2) \implies \cotan(\pi x_1) = \cotan(\pi x_2)$

$\implies \pi x_1 = \pi x_2 + k\pi$ for some
integer $k$.

$$\Rightarrow x_1 = x_2 + k \quad \text{for some integer } k$$

$$0 < x_1 < 1 \quad \Rightarrow \quad \lfloor x_1 \rfloor = 0 \quad \Rightarrow \quad k = 0$$

$$0 < x_2 < 1 \quad \Rightarrow \quad k < x_2 + k < k + 1 \quad \Rightarrow \quad x_1 = x_2$$

$$\Rightarrow \quad \lfloor x_2 + k \rfloor = k$$

$$x_1 = x_2 + k$$

**Onto.** From calculus, you know that

1. $f$ is continuous. \(\Rightarrow\) by the mean value theorem
2. $$\lim_{x \to 1^-} f(x) = -\infty$$ we have that
3. $$\lim_{x \to 0^+} f(x) = +\infty$$ \(\forall c \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = c\)

It is OK if a student refers to the graph of this function to show surjectivity.