1. Clearly there is a bijection between the possible paths and elements of \( \times 1, 2, 3, 4 \times a, b, c, d, e \). So by the multiplication principle, the number of possible paths is equal to \( (4)(5) = 20 \).

2. We prove that \( |A_1 \times A_2 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k| \) if \( |A_i| < \infty \) by induction on \( k \).

**Base** \( k = 1 \) is clear.

**Inductive step.** \( |A_1 \times \ldots \times A_k| = \prod_{i=1}^{k} |A_i| \Rightarrow |A_1 \times \ldots \times A_k \times A_{k+1}| = \prod_{i=1}^{k+1} |A_i| \).

**Proof of the inductive step.**

**Claim.** \( f : (A_1 \times \ldots \times A_k) \times A_{k+1} \to A_1 \times \ldots \times A_{k+1} \)

\[ f((a_1, \ldots, a_k), a_{k+1}) = (a_1, \ldots, a_{k+1}) \]

is a bijection.

**Proof of the claim.** \( f \) is 1-1.

\[ f((a_1, \ldots, a_k), a_{k+1}) = f((a_1', \ldots, a_k'), a_{k+1}') \]

\[ \Rightarrow (a_1, \ldots, a_{k+1}) = (a_1', \ldots, a_{k+1}') \]

\[ \Rightarrow a_1 = a_1', \quad a_2 = a_2', \quad \ldots, \quad a_{k+1} = a_{k+1}' \]

\[ \Rightarrow (a_1, \ldots, a_k, a_{k+1}) = (a_1', \ldots, a_k', a_{k+1}') \]
• \( \mathcal{A} \) is onto.

For any \((a_1, \ldots, a_{k+1})\) we have \(\mathcal{A}((a_1, \ldots, a_k), a_{k+1}) = (a_1, \ldots, a_k)\) 

Hence by the above claim we have

\[
|A_1 \times \cdots \times A_{k+1}| = |(A_1 \times \cdots \times A_k) \times A_{k+1}|
\]

\[
= |A_1 \times \cdots \times A_k| \cdot |A_{k+1}| \quad \text{(by the mult. principle)}
\]

\[
= (|A_1| \cdot |A_2| \cdots \cdot |A_k|) \cdot |A_{k+1}| \quad \text{(induction hypothesis)}
\]

\[
= \prod_{i=1}^{k+1} |A_i|.
\]

3, 4. The main idea behind the solutions of 3 and 4:

Claim. There is a bijection between the set

\[\text{Fun}(\mathbb{N}, \mathbb{N}, B)\]

of functions from \(\mathbb{N}, \mathbb{N}\) and \(B\) and

\[B \times B \times \cdots \times B\]

\(n\) copies

Proof of the claim.

Let \(g : \text{Fun}(\mathbb{N}, \mathbb{N}, B) \rightarrow B \times B \times \cdots \times B\),

\[g(f) := (f(1), f(2), \ldots, f(n)).\]
5. We use inclusion-exclusion:

\[ |X\setminus(A\cup B\cup C)| = |X| - |A| - |B| - |C| + |A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C|. \]

The following is the key fact:

Claim. For any positive integers \( d \) and \( n \), if \( d \mid n \), then

\[ |\{ k \in \mathbb{Z} \mid 1 \leq k \leq n, \ d \mid k \}| = \frac{n}{d}. \]

**Proof of Claim.** We show that

\[ f: \{1, 2, \ldots, n/d\} \rightarrow \{ k \in \mathbb{Z} \mid 1 \leq k \leq n, \ d \mid k \} \]

\[ f(i) = di \]

is a bijection.

1. **\( f \) is well-defined.**
   - \( 1 \leq i \leq n/d \Rightarrow d \leq di \leq n \Rightarrow 1 \leq di \leq n \)
   - \( d \mid di \).

2. **\( f \) is 1-1.**
   \[ f(i_1) = f(i_2) \Rightarrow di_1 = di_2 \Rightarrow i_1 = i_2. \]

3. **\( f \) is onto.**
   \[ d \mid k \Rightarrow k = di \text{ for some } i \in \mathbb{Z} \]
\[1 \leq k \leq n \implies 1 \leq \phi(k) \leq n\]

\[\implies \frac{1}{d} \leq \phi(k) \leq \frac{n}{d} \quad \text{for some } i \in \mathbb{Z} \ni \frac{1}{d} > 0\]

\[\implies k = \phi(i) \quad \text{for some } i \in \mathbb{Z} \ni \phi(i) \leq \frac{n}{d} .\]

Using the above claim:

\[|A| = \frac{300}{2} = 150\]

\[|B| = \frac{300}{3} = 100\]

\[|C| = \frac{300}{5} = 60\]

\[A \cap B = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 300, \ 6 \mid k \} \]

\[A \cap C = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 300, \ 10 \mid k \} \]

\[B \cap C = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 300, \ 15 \mid k \} \]

\[A \cap B \cap C = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 300, \ 30 \mid k \} \]

\[\implies |A \cap B| = \frac{300}{6} = 50\]

\[|A \cap C| = \frac{300}{10} = 30\]

\[|B \cap C| = \frac{300}{15} = 20\]

\[|A \cap B \cap C| = \frac{300}{30} = 10\]

\[\implies |X \setminus (A \cup B \cup C)| = 300 - 150 - 100 - 60 + 50 + 30 + 20 - 10\]

\[= 80\]
6. By Pigeonhole principle at least two points $p_i$ and $p_j$ share "the same pigeonhole" $a$, $b$, $c$ or $d$.

$$|p_i - p_j| \leq \text{diameter of a } (\frac{1}{2}) \times (\frac{1}{2}) \text{ square}$$

$$= \frac{1}{\sqrt{2}}.$$