1.(a) False. For instance, let $A=\{1\}$ and $B=\{1,\{1\}\}$.

Then $A\subseteq B$ and $A\in B$.

(b) True. $\{1,\{1,2\},\varnothing\}$ has three elements and

one of them is $\{1,2\}$, and

$\{\varnothing\}$ has a single element which is the empty set.

(c) False. For instance $2\in\{1,2,3\}\wedge\{1,2,3\}\cap\{2,2\}=\{2\}$

$\wedge\ 2\neq 1$.

[Remark. Show that $a\in\{1,2,3\}\wedge\{1,2,3\}\cap\{a,2\}=\{2\}$

implies $a=2$.]

[Think about the following: Is it true or false?

$a\in\{1,3\}\wedge\{1,2,3\}\cap\{a,2\}=\{2\}\Rightarrow a=1$.]

(d) True. If not, then $1\in\{1,2,3\}$ and $1\notin\{a,2\}$

which implies $1\in\{1,2,3\}\setminus\{a,2\}=\{3\}$. That is a

contradiction. ■

[Remark. The converse is also true: $\{1,2,3\}\setminus\{1,2\}=\{3\}$.]

2.(a) If there is a set A s.t. $A\in A$, let $X=\{A\}$.

By the axiom of regularity, $\exists Y\in X,\ Y\cap X=\varnothing$.

Since A is the only element of X, we have

$$A \cap X = \emptyset.$$

On the other hand, $A \in A$ and $A \in X$. So $A \in A \cap X$, which is a contradiction.

(b) Suppose to the contrary that there are sets A and B s.t.

$$A \in B \quad \wedge \quad B \in A.$$

Let $X = \{A, B\}$. Hence $A \in B \cap X \ \wedge B \in A \cap X$. So $A \cap X \neq \emptyset \ \wedge \ B \cap X \neq \emptyset$. Since A and B are the only elements of X, we have $\forall Y \in X,\ Y \cap X \neq \emptyset$; which contradicts the axiom of regularity. $\blacksquare$

3. (a) False. Let X be a set with at least two distinct element $a \neq b$. Then $\{a, b\} = \{b, a\} \wedge a \neq b$.

(b) True. $\{a, \{a, b\}\} = \{c, \{c, d\}\} \overset{?}{\Longrightarrow} a = c \wedge b = d$.

<u>First Step</u>. $a = c$.

<u>Pf of the first step</u>. Suppose to the contrary $a \neq c$.

$$\left.\begin{array}{l} a\in\{a,\{a,b\}\} \\ \{a,\{a,b\}\}=\{c,\{c,d\}\} \end{array}\right\} \Rightarrow \left.\begin{array}{c} a=c \vee a=\{c,d\} \\ a\neq c \end{array}\right\} \Rightarrow \mathbf{a=\{c,d\}} \quad \mathrm{I}$$

$$\left.\begin{array}{l} c\in\{c,\{c,d\}\} \\ \{a,\{a,b\}\}=\{c,\{c,d\}\} \end{array}\right\} \Rightarrow \left.\begin{array}{c} c=a \vee c=\{a,b\} \\ a\neq c \end{array}\right\} \Rightarrow \mathbf{c=\{a,b\}} \quad \mathrm{II}$$

$\mathrm{I}, \mathrm{II} \Rightarrow \quad a\in c \wedge c\in a$ which is a contradiction by 2(b).

<u>Second Step</u> $b=d$.

<u>Pf of the second step.</u>

$$\left.\begin{array}{l} \{a,b\}\in\{a,\{a,b\}\} \\ \{a,\{a,b\}\}=\{a,\{a,d\}\} \end{array}\right\} \Rightarrow \{a,b\}=a \vee \{a,b\}=\{a,d\} \quad \mathrm{III}$$

Notice that $\{a,b\}=a \Rightarrow a\in a$ which contradicts 2(a).

Hence $\{a,b\}\neq a$. Thus by III $\{a,b\}=\{a,d\}$.

$\{a,b\}=\{a,d\} \Rightarrow b=a \vee b=d$.

If $b=a$, then $d\in\{a,a\}=\{a\}$ which implies $d=a=b$.

So in either case $b=d$. ∎

4. (a) $\exists\, \varepsilon>0, \forall\, \delta>0, \exists\, x\in\mathbb{R},\ |x-1|<\delta \wedge |x^2-1|\geq\varepsilon$.

(b) $\exists\, \varepsilon>0, \exists\, x\in\mathbb{R}, \forall\, n\in\mathbb{Z},\ |x-n|\geq\varepsilon$.

(c) $\exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall m,n \in \mathbb{Z}, |x - m - n\alpha| \geq \varepsilon$.

(This is an acceptable answer. Here we assumed α is a given fixed irrational number, but the better answer (which is different) is the following.)

$\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}, \exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall m,n \in \mathbb{Z}, |x - m - n\alpha| \geq \varepsilon$.

(d) $\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}^{>0},$

1. $(\forall q, r \in \mathbb{Z},\ a \neq bq + r \vee r \geq b \vee r < 0)$

$\vee$

2. $(\exists q_1, q_2, r_1, r_2 \in \mathbb{Z},\ a = bq_1 + r_1 \wedge a = bq_2 + r_2 \wedge 0 \leq r_1 < b \wedge 0 \leq r_2 < b \wedge (q_1 \neq q_2 \vee r_1 \neq r_2))$

In plain English, we are saying there is no unique such pair (q,r) of integers if either there is NO such pairs (1) or there are more than one pair (2).

5. (a) True. $\forall y \in \mathbb{R},\ y^2 \geq 0 > 2013 + (-2014)$

So $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ y^2 > 2013 + x$.

(b) False. We have to show $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ y^3 \leq 2013 + x$

If $2013+x \geq -1$, then let $y=-1 \Rightarrow -1^3=-1 \leq 2013+x$.

If $2013+x<-1$, then let $y=2013+x$

$\Rightarrow$ $y^3=(2013+x)^3=(2013+x)^2(2013+x)\}\Rightarrow$

and $(2013+x)^2>1$

$y^3< 2013+x$.

(c) True. Construct the proof backwards.

$$\frac{1000}{n}<\varepsilon \Leftarrow \frac{1000}{\varepsilon}<n \qquad (\text{Since } \varepsilon>0)$$

$$\Leftarrow N\leq n \wedge \frac{1000}{\varepsilon}<N \quad \text{I}$$

As it is given in the hint, $\forall x\in\mathbb{R}, \exists N\in\mathbb{Z}, x<N$.

So $\forall \varepsilon>0, \exists N\in\mathbb{Z}^+, \frac{1000}{\varepsilon}<N$. Thus by I

$$\forall\varepsilon>0, \exists N\in\mathbb{Z}^{>0}, \quad N\leq n \Rightarrow \frac{1000}{n}<\varepsilon.$$

6. (a) $\forall\delta>0, \exists n\in\mathbb{Z}^+, \frac{1}{2n\pi}<\delta \wedge \frac{1}{(2n+1)\pi}<\delta$. (?)

Construct the proof backwards

$$\frac{1}{2n\pi}<\delta \wedge \frac{1}{(2n+1)\pi}<\delta \Leftarrow \frac{1}{\pi\delta}<2n<2n+1$$

$$\Leftarrow \frac{1}{\pi\delta}<n. \quad \text{I}$$

As it is said in the hint of problem 5,

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, \quad x < n.$$

So $\forall \delta > 0, \exists n \in \mathbb{Z}^+, \ \frac{1}{\pi\delta} < n$. Hence by I

$$\forall \delta > 0, \exists n \in \mathbb{Z}^+, \ \frac{1}{2n\pi} < \delta \wedge \frac{1}{(2n+1)\pi} < \delta.$$

(b) Suppose to the contrary that there is L such that

$$\forall \varepsilon > 0, \exists \delta > 0, \ |x| < \delta \Rightarrow \left|\cos\left(\frac{1}{x}\right) - L\right| < \varepsilon. \quad \text{II}$$

So in particular for $\varepsilon = 1/4$ there is $\delta > 0$ s.t. II holds, i.e.

$$\exists \delta > 0, \quad |x| < \delta \Rightarrow \left|\cos\left(\frac{1}{x}\right) - L\right| < 1/4. \quad \text{III}$$

For a given $\delta > 0$ s.t. III holds by part (a) there is $n \in \mathbb{Z}^+$ s.t. $0 < \frac{1}{2n\pi} < \delta \ \wedge 0 < \frac{1}{(2n+1)\pi} < \delta$. IV

Hence by III and IV we have

$$\left|\cos\left(1/\left(\frac{1}{2n\pi}\right)\right) - L\right| < 1/4 \wedge \left|\cos\left(1/\left(\frac{1}{(2n+1)\pi}\right)\right) - L\right| < 1/4.$$

Since $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$, we have

$$|1 - L| < 1/4 \wedge |-1 - L| < 1/4.$$

Thus $\quad 1/2 = 1/4 + 1/4 > |-1-L| + |1-L| \geq |(1-L)-(-1-L)| = 2$

which is a contradiction.

Alternative logical approach:

$$\nexists L \in \mathbb{R},\ \forall \varepsilon > 0, \exists \delta > 0,\ 0 < |x| < \delta \Rightarrow \left|\cos\left(\frac{1}{x}\right) - L\right| < \varepsilon$$

$$|||$$

$$\forall L \in \mathbb{R},\ \exists \varepsilon > 0, \forall \delta > 0, \exists x,\ 0 < |x| < \delta \wedge \left|\cos\left(\frac{1}{x}\right) - L\right| \geq \varepsilon.$$

In the above argument, we are proving that $\varepsilon = 1/4$ works, i.e.

$$\forall L \in \mathbb{R},\ \forall \delta > 0,\ \exists x,\ 0 < |x| < \delta \wedge \left|\cos\left(\frac{1}{x}\right) - L\right| \geq 1/4.$$

Part (a): $\forall \delta > 0, \exists n \in \mathbb{Z}^+,\ 0 < \frac{1}{2\pi n}, \frac{1}{(2n+1)\pi} < \delta$

$$\Rightarrow \quad \forall \delta > 0, \exists x_1, x_2,\ 0 < x_1, x_2 < \delta \wedge \cos\left(\frac{1}{x_1}\right) = 1$$

$$\wedge \cos\left(\frac{1}{x_2}\right) = -1. \quad \text{I}$$

(Since $\cos(2\pi n) = 1 \wedge \cos((2n+1)\pi) = -1$.)

On the other hand, $\forall L \in \mathbb{R},\ |1-L| \geq 1/4 \vee |-1-L| \geq 1/4$. II

(If not, then $1/2 = 1/4 + 1/4 > (1-L) - (-1-L) = 2$, which is a contradiction.)

I, II $\Rightarrow$

$$\forall L \in \mathbb{R}, \forall \delta > 0,\ \exists x,\ 0 < x < \delta \quad \wedge \left|\cos\left(\frac{1}{x}\right) - L\right| \geq 1/4. \quad \blacksquare$$