1. Find all the solutions of

\[ \begin{bmatrix} 14 \\ 21 \end{bmatrix} \begin{bmatrix} x \\ 21 \end{bmatrix} = \begin{bmatrix} 28 \\ 21 \end{bmatrix}. \]

2. Find all the solutions of

\[ x \equiv 5 \pmod{8}, \]
\[ x \equiv 7 \pmod{9}. \]

3. Let \((G, \cdot)\) be a group. Let \(e\) be the identity element of \(G\). Suppose for any \(a \in G\) we have

\[ a^2 = e. \]

Prove that \(G\) is abelian, i.e. \(ab = ba\) for any \(a, b \in G\).

4. Let \(n \in \mathbb{Z}^\geq 2\). Prove that for any \([a]_n\)

either \(\exists [b]_n \text{ s.t. } [a]_n [b]_n = [1]_n\)

or \(\exists [b]_n \text{ s.t. } [a]_n [b]_n = [0]_n.\)

(Any element is either a unit or a zero-divisor.)
5. Let \( p \) be a prime number. Then we know

\[
\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}.
\]

(i) Prove that \( \left[ a \right]_p = \left[ a \right]_p^{-1} \iff \left[ a \right]_p = [\pm 1]_p \).

(ii) Prove that \( \left[ 1 \right]_p \cdot \left[ 2 \right]_p \cdot \ldots \cdot [p-1]_p = [-1]_p \)

(This is equivalent to \( (p-1)! \equiv -1 \pmod{p} \).

**Hint for (ii)** Pair each term with its modular inverse and notice that by part (i) you can actually do it unless the term is either \( [1]_p \) or \( [p-1]_p \).

6. Let \( n \in \mathbb{Z}^{\geq 1} \). For a divisor \( d \) of \( n \), let

\[
A_d := \{ k \in \mathbb{Z} \mid 1 \leq k \leq n, \ \gcd(k,n) = d \}.
\]

(i) Prove that \( |A_d| = \varphi(\frac{n}{d}) \).

[Recall. \( \varphi(m) = |\mathbb{Z}_m^*| = \left| \frac{\mathbb{Z}}{\mathbb{Z} \cap d\mathbb{Z}} \right| \).]

**Hint.** \( \gcd(k,n) = d \iff \gcd\left( \frac{k}{d}, \frac{n}{d} \right) = 1 \).

(ii) Prove that \( \sum_{d|n} \varphi(\frac{n}{d}) = n \).
Hint. Notice that \( \bigcup_{d|n} A_d \cap A_{d_1} \cap A_{d_2} = \emptyset \) if \( d_1 \neq d_2 \).

7. Let \( p \) be a prime number. Prove that, for any \( a, b \in \mathbb{Z} \), we have

\[
\left( [a]_p + [b]_p \right)^p = [a]_p + [b]_p.
\]

(Hint. One approach is to use the binomial expansion:

\[
(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\]

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1) \cdots (n-i+1)}{i!} \). And then show \( p \mid \binom{p}{i} \) if \( p \) is prime and \( 1 \leq i \leq p-1 \).)