The fifth problem set.

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11:04 PM

1. Let $(G,\cdot)$ be a group, and $g\in G$. Let

$$C_G(g) := \{ h\in G \mid g.h = hg \}.$$

Prove that $C_G(g)$ is a subgroup of G.

($C_G(g)$ is called the centralizer of g.)

2. Let $(G,\cdot)$ be a group, and $H\leq G$. Let

$$N_G(H) := \{ g\in G \mid gHg^{-1} = H \}.$$

Prove that $N_G(H)$ is a subgroup of G.

($N_G(H)$ is called the normalizer of H.)

3. Let $(G,\cdot)$ be a group, and $H_1, H_2 \leq G$.

Prove that $H_1 \cup H_2 \leq G \Longleftrightarrow (H_1 \subseteq H_2$ or $H_2 \subseteq H_1)$.

4. (a) Prove that S_3 is NOT cyclic.

(Hint. First observe that any cyclic group is abelian, i.e. $a, b \in G \Rightarrow ab = ba$.)

(b) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT cyclic.

(Hint. For any $(x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$(x,y) + (x,y) = ([0]_2, [0]_2).$$

$\Rightarrow$ $o((x,y)) \leq 2$.)

5. Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m,n) = 1$.

(Hint ($\Rightarrow$) Show that $o((x,y)) \mid \operatorname{lcm}(m,n)$ for any m and n, and any $(x,y) \in \mathbb{Z}_m \times \mathbb{Z}_n$.

($\Leftarrow$) Use Chinese Remainder Theorem to show

$$\mathbb{Z}_m \times \mathbb{Z}_n = \langle ([1]_m, [1]_n) \rangle$$

if $\gcd(m,n) = 1$.)

6. Find a group G and $a, b \in G$ such that

$$o(a) < \infty, \quad o(b) < \infty, \quad \text{and} \quad o(ab) = \infty.$$

(Hint. There are lots of such examples. Here is one such examples: think about isometries of the real line. They are either a reflection about a point or a translation.)

7. Let $(G,\cdot)$ be a finite group. Suppose for any positive integer n,

$$|\{g\in G \mid g^n=e\}| \le n.$$

(a) For any d, let $B_d := \{g\in G \mid o(g)=d\}$.

Prove that, if $B_d \neq \emptyset$, then $|B_d| = \varphi(d)$

(Hint. Use $o(g^m) = \dfrac{o(g)}{\gcd(m, o(g))}$.)

(b) Suppose $\forall g\in G$, $g^{|G|}=e$. Prove that G is cyclic.

(Remark. In lectures we will see that the mentioned assumption, $g^{|G|}=e$, always holds.

Hint. By the mentioned assumption $o(g) \mid |G|$

for any $g \in G$. So $G = \bigcup_{d \mid |G|} B_d$ and

$B_{d_1} \cap B_{d_2} = \emptyset$ if $d_1 \neq d_2$. Hence

$$|G| = \sum_{d \mid |G|} |B_d| \underset{\text{part (a)}}{\leq} \sum_{d \mid |G|} \varphi(d) \underset{\text{last week's problem}}{=} |G|.$$

Equality holds $\Rightarrow \forall\, d \mid |G|$ we have

$$|B_d| = \varphi(d).$$

In particular, $|B_{|G|}| = \varphi(|G|) \neq 0$.)