The seventh problem set.

1. Let $\sigma = c_1 \cdot c_2 \cdot \ldots \cdot c_s$ and $c_i$'s be disjoint cycles. Suppose $c_i = (a_{i1}, a_{i2}, \ldots, a_{ik_i})$ and $k_i \geq 2$. Prove that $o(\sigma) = \text{lcm}(k_1, \ldots, k_s)$.

(Hint: In class we considered the natural action of $\langle \sigma \rangle$ on $\mathbb{Z}/1, 2, \ldots, n^{\mathbb{G}}$ and proved that the orbit of $a_{i1}$ under this action is $\mathbb{Z}/a_{i1}, \ldots, a_{i1}k_i, \ldots$.)

2. $c_i \cdot c_j = c_j \cdot c_i \Rightarrow \sigma^l = c_1^l \cdot \ldots \cdot c_s^l$ for any integer $l$.)

2. Prove that $\langle (1,2), (2,3), \ldots, (n-1,n) \rangle = S_n$.

(Hint. We have proved that any permutation is a product of transpositions, i.e.

$\langle (i,j) \mid 1 \leq i < j \leq n \rangle = S_n$.

2. We proved $(i+1, i+2)(i+2, i+3) \ldots (j-1, j) = (i+1, i+2, \ldots, j)$)
3. Prove that $\langle (1, 2), (1, 2, \ldots, n) \rangle = S_n$

(Hint: Consider $(1, 2, \ldots, n)^2 (1, 2) (1, 2, \ldots, n)^{-1}$

and use problem 2.)

Use problem 1, to answer the following questions.

4. (a) Show that an element of order 5 in $S_9$ is a 5-cycle. Conclude that $S_9$ has $9 \times 8 \times 7 \times 6$ many elements of order 5.

(b) Show that an element of order 5 in $S_{10}$ is either a 5-cycle or a product of two disjoint cycles. Find the number elements of order 5 in $S_{10}$.

5. Show that $o(ab a^{-1}) = o(b)$ and $o(ab) = o(ba)$.

**Conjugacy Classes of $S_n$.**
Any permutation $\sigma \in S_n$, as we proved in the class, can be uniquely written as a product of disjoint cycles $c_i$. Suppose length of $c_i$ is $k_i$ and

$$\sigma = c_1 \cdot c_2 \cdot \ldots \cdot c_l, \quad 2 \leq k_1 \leq k_2 \leq \ldots \leq k_l.$$ 

The cyclic type of $\sigma$ is $1, \ldots, 1, k_1, k_2, \ldots, k_l$. 

For instance, the cyclic type of (1) is

$$1, 1, \ldots, 1$$

$n$ times

The cyclic type of

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
4 & 2 & 6 & 7 & 6 & 5 & 1
\end{array}$$

is

$$1 \rightarrow 4 \rightarrow 7 \quad 2 \quad 3 \rightarrow 6 \rightarrow 5$$

So the cyclic type is $1,3,3$.

The cyclic type of $(1,2,3) \in S_7$ is

$$1,1,1,1,3$$

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\overleftarrow{1} & \overleftarrow{2} & \overleftarrow{3} & \overrightarrow{4} & \overrightarrow{5} & \overrightarrow{6} & \overrightarrow{7}
\end{array}$$
6. Let $\tau \in S_{n_{2015}}$ and $\sigma = (1, 2, 3)(3, 4)(6, 7)$. Find the cyclic types of $\sigma$ and $\tau \sigma \tau^{-1}$.

(Hint: $\tau \sigma_1 \sigma_2 \tau^{-1} = (\tau \sigma_1 \tau^{-1})(\tau \sigma_2 \tau^{-1})$.

Remark. You can see that the same argument shows that $\sigma$ and $\tau \sigma_1 \tau^{-1}$ have the same cyclic type for any $\sigma, \tau \in S_n$.

7. Show that $\exists \tau \in S_{12}$ s.t. $\sigma_2 = \tau \sigma_1 \tau^{-1}$

where $\sigma_1 = (1, 2)(3, 4, 5)(6, 7)$

and $\sigma_2 = (1, 3)(6, 10, 12)(8, 9)$.

Remark. You can see that the same argument shows that, if $\sigma_1$ and $\sigma_2$ have the same cyclic type, then $\exists \tau \in S_n$, $\sigma_2 = \tau \sigma_1 \tau^{-1}$.

Remark. By definition, you can see that $1 \leq m_1 \leq m_2 \leq \ldots \leq m_k$ is a cyclic type of an element of $S_n$ if and
only if \( m_1 + m_2 + \ldots + m_k = n \).

The number of ways \( n \) can be written as a sum of increasing positive integers is denoted by \( p(n) \). For instance,

\[
\begin{align*}
p(1) &= 1 \\
p(2) &= 2 \quad 1+1 \text{ and } 2 \\
p(3) &= 3 \quad 1+1+1, \ 1+2, \ 3 \\
p(4) &= 5 \quad 1+1+1+1, \ 1+1+2, \ 1+3, \ 2+2,
\end{align*}
\]

By the above remarks, you can see that

\[ \text{the number of conjugacy classes of } S_n \]

is equal to \( p(n) \).