The ninth problem set.

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8:30 PM

1.(a) Show that any group of prime order is cyclic.

(Hint. $|G|=p$ and $e \neq g \in G$. What can $o(g)$ be?)

(b) Let G be a cyclic group, and $\phi: G \to H$ be a group homomorphism. Show that $\mathrm{Im}\,\phi$ is cyclic.

(Hint. $G = \{g^i \mid i \in \mathbb{Z}\} \Rightarrow \mathrm{Im}\,\phi = \{\phi(g^i) \mid i \in \mathbb{Z}\}$.)

2. Suppose G is a group and $|G|=6$. Suppose $\exists\, a, b \in G$ s.t. $o(a)=2$ and $o(b)=3$.

(i) If $ab=ba$, then show that $G \simeq \mathbb{Z}_6$.

(It is enough to show it is cyclic.)

(ii) If $ab \neq ba$, then show that $ab = b^{-1}a$

(Hint. Use $G/\langle b \rangle$ to show

$$G = \{e, b, b^2, a, ab, ab^2\}.)$$

(iii) Show that $G \simeq S_6$ if $ab \neq ba$.

3. (i) Show that any element of $\mathbb{Q}/\mathbb{Z}$ is torsion.

(Example. $2\left(\frac{1}{2}+\mathbb{Z}\right) = 1+\mathbb{Z} = \mathbb{Z}$.)

(ii) Find $o\left(\frac{1}{n}+\mathbb{Z}\right)$, and conclude $|\mathbb{Q}/\mathbb{Z}| = \infty$.

(iii) Show that $\left\langle \frac{a_1}{b_1}+\mathbb{Z}, \frac{a_2}{b_2}+\mathbb{Z} \right\rangle \simeq \mathbb{Z}_{\operatorname{lcm}(b_1,b_2)}$

if $\gcd(a_1,b_1) = \gcd(a_2,b_2) = 1$.

(Hint. ① $\gcd(a,b)=1 \overset{?}{\Rightarrow} \left\langle \frac{a}{b}+\mathbb{Z}\right\rangle = \left\langle \frac{1}{b}+\mathbb{Z}\right\rangle$

② $$\frac{1}{\operatorname{lcm}(b_1,b_2)} = \frac{\gcd(b_1,b_2)}{b_1 b_2}$$

$$= \frac{r\,b_1 + s\,b_2}{b_1 b_2} \quad \text{for some } r,s\in\mathbb{Z}$$

$$= \frac{r}{b_2} + \frac{s}{b_1}$$

$$\Rightarrow \frac{1}{\operatorname{lcm}(b_1,b_2)} + \mathbb{Z} \in \left\langle \frac{1}{b_1}+\mathbb{Z}, \frac{1}{b_2}+\mathbb{Z} \right\rangle.)$$

[Remark. In fact, by a similar argument one can see that any finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.]

4. Prove that $\mathbb{Z}\times\mathbb{Z}/\langle(a,b)\rangle \simeq \mathbb{Z}$ if and only if

$\gcd(a,b) = 1$.

(Hint. ($\Leftarrow$) Similar to the examples presented in the lecture, give $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ s.t. $\ker \phi = \langle (a,b) \rangle$ and $\operatorname{Im} \phi = \mathbb{Z}$. (How about $bx - ay$?)

($\Rightarrow$) Similar argument as given in the lecture.)

5.(i) Show that $\mathbb{Z}_{nk} / n\mathbb{Z}_{nk} \simeq \mathbb{Z}_n$.

(ii) Show that $\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle ([2]_4, [2]_6) \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(Hint: $([x]_4, [y]_6) \mapsto ([x]_2, [y]_2)$

$\ker \phi = \{ (2[x]_4, 2[y]_6) \mid x, y \in \mathbb{Z} \}$

$= \langle ([2]_4, [0]_6), ([0]_4, [2]_6) \rangle$

$\overset{?}{=} \langle ([2]_4, [2]_6) \rangle$.

To see this equality, you can compare their sizes.)

6. Show that $G/Z(G)$ cannot be a non-trivial cyclic group.

(i.e. $G/Z(G) = \langle g_0 Z(G) \rangle \overset{?}{\Longrightarrow} G$ is abelian.

$\Rightarrow G = \{ g_0^i z \mid i \in \mathbb{Z} \text{ and } z \in Z(G) \}$.)

(Recall that $Z(G) = \{ g \in G \mid \forall g' \in G,\ gg' = g'g \}$.)

7. Let $N = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.

(i) Show that $N \trianglelefteq S_4$.

(ii) Show that $S_4/N \simeq S_6$. (Hint. Use problem 2)