Lecture 17: orbits and the action of cyclic groups

Proposition \( G \subset \mathcal{X} \) and \( x_0 \in \mathcal{X} \). Then

(i) \( G_{x_0} := \{ g \in G \mid g \cdot x_0 = x_0 \} \) is a subgroup.

(ii) \( \theta: G / G_{x_0} \rightarrow O(x_0), \quad \theta(g G_{x_0}) = g \cdot x_0 \)

is a well-defined bijection.

Proof. (i) \( e \cdot x_0 = x_0 \Rightarrow e \in G_{x_0} \Rightarrow G_{x_0} \neq \emptyset \)

So by Subgroup Criteria we have to check the following

\[ g_1, g_2 \in G_{x_0} \Rightarrow g_1^{-1} g_2 \in G_{x_0} \]

\[ g_1 \cdot x_0 = x_0 \Rightarrow g_1 \cdot x_0 = g_2 \cdot x_0 \Rightarrow (g_1^{-1} g_2) \cdot x_0 = x_0 \]

\[ g_2 \cdot x_0 = x_0 \]

\[ \Rightarrow g_1^{-1} g_2 \in G_{x_0} \]

(ii) well-defined.

\[ g_1 G_{x_0} = g_2 G_{x_0} \Rightarrow g_1 = g_2 h \text{ for some } h \in G_{x_0} \]

\[ \Rightarrow g_1 \cdot x_0 = (g_2 h) \cdot x_0 \]

\[ \Rightarrow g_1 \cdot x_0 = g_2 \cdot (h \cdot x_0) \]
\[ g_1 \cdot x_0 = g_2 \cdot x_0 \Rightarrow (g_1^{-1} g_2) \cdot x_0 = x_0 \]
\[ \Rightarrow g_1^{-1} g_2 = h \in G_{x_0} \]
\[ \Rightarrow g_2 = g_1 h \in g_1 G_{x_0} \]
\[ \Rightarrow g_2 G_{x_0} = g_1 G_{x_0}. \]

\text{Onto} \quad \text{It is clear from the definition of } O(x_0). \]

\text{Cor.} If \( G \) is a finite group, then
\[ |O(x_0)| = |G : G_{x_0}| \mid |G|. \]

\text{Pf.} By the previous Proposition, \( |O(x_0)| = |G/G_{x_0}| \mid |G| \)
which is \( |G : G_{x_0}| \) by definition. And we have already proved \( |G| = |G_{x_0}| \cdot [G : G_{x_0}] \Rightarrow \]
\[ |O(x_0)| \mid |G|. \]

Since the set of left cosets is of particular importance, let's summarize its properties:
• \( g_1 H = g_2 H \iff g_1 g_2^{-1} \in H \).

• \( H g_1 = H g_2 \iff g_1 g_2^{-1} \in H \).

How does a cyclic group act on a set?

Let's assume \( \langle a \rangle \) is a finite group of order \( d \).

Suppose \( \langle a \rangle \cap X \). How does orbits "look like"?

\[
\begin{align*}
x_0 & \rightarrow a.x_0 & \rightarrow a^2.x_0 & \rightarrow a^3.x_0 & \rightarrow \ldots
\end{align*}
\]

At some point we should come back as \( a^d = e \)
and so \( a^d.x_0 = x_0 \). And so we get a cycle.

1. Size of this cycle divides \( d \).

2. Either this cycle is the entire \( X \),
or take \( x_1 \) in \( X \) outside this cycle
and repeat.

So \( X \) is disjoint union of bunch of cycles
(whose size divides \( d \)) and \( a \) just "rotates" points.
On these cycles.

Schreier directed graphs: \( G = \langle S \rangle \cup X \)

Vertices = \( X \)

\((x_1, x_2)\) is an edge if \( \exists s \in S \) s.t. \( x_2 = s \cdot x_1 \)

So in the case of finite cyclic group we get the above cycles.

We also discussed the following examples:

1. \( S_n \cup \{1, 2, \ldots, n\} \).

\( G_n := \) stabilizer of \( n \)

\[ = \{ \sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \mid \sigma(n) = n \} \]

So \( |G_n| = (n-1)! \). \( G_n \) is more or less \( S_{n-1} \).

\( O(n) = \{1, 2, \ldots, n\} \).
\[ [S_n : G_n] = \frac{|S_n|}{|G_n|} = \frac{n!}{(n-1)!} = |O(n)|. \]

2. \( G \trianglelefteq G \) by conjugation, i.e.

\[ g \cdot g' := gg'g^{-1}. \]

- \( O(g') = \{ gg'g^{-1} \mid g \in G \} = C_G(g') \)
  is called the conjugacy class of \( g' \).

- \( gg'g^{-1} \) is called a conjugate of \( g' \).

- Stabilizer of \( g' = \{ g \in G \mid gg'g^{-1} = g' \} \)
  \[ C_G(g') = \{ g \in G \mid gg' = g'g \} \]
  is called the centralizer of \( g' \) in \( G \).

So we have \( |C_G(g')| = [G : C_G(g')] \).