In the previous lecture we observed that

\[ \langle a \rangle \triangleleft X \quad \text{and} \quad |\langle a \rangle| = o(a) \]

\[ \text{disjoint union of cycles s.t.} \]

\[ \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \]

1. Each cycle consists of one orbit.
2. Length of each cycle divides \( |\langle a \rangle| = o(a) \).
3. \( a \) acts by rotating 4-step on each cycle.

For instance, if \( o(a) = p \), then all the cycles are either of size 1 (fixed points) or of size \( p \).

One important example is the action of \( \langle a \rangle \) on \( \mathbb{Z}/p \mathbb{Z} \), where \( \sigma \in S_n \). Let's consider the following element of \( S_6 \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
4 & 3 & 2 & 6 & 5 & 1 \\
\end{array}
\]

\[ \text{cycle of length } 3 \quad \text{cycle of length } 2 \quad \text{cycle of length } 1 \]

So we can understand \( \sigma \) by looking at these cycles.

\[ \forall \sigma \in S_n, \text{ let } \text{Fix}(\sigma) = \{i : \sigma(i) = i\}. \]

These give us cycles of length 1.

We say \( \sigma \) is a cycle of length \( k \) if its Schreier graph consists of one cycle of length \( k \) and bunch of cycles of length 1.

\[
\begin{array}{cccccc}
1 & 4 & 6 & 2 & 3 & 5 \\
\circ & \circ & \circ & \circ & \circ \\
1 & 6 & 2 & 5 & \text{Cycle of length } 3 \quad \text{Cycle of length } 4
\end{array}
\]
Exp. \[ c_1 \xrightarrow{4} 6 \xrightarrow{2} 3 \xrightarrow{5} \quad \text{cycle of length 3} \]
\[ c_2 \xrightarrow{1} 4 \xrightarrow{6} \xrightarrow{3} 5 \quad \text{cycle of length 2} \]

What are \( c_1 \circ c_2 \) and \( c_2 \circ c_1 \)?

\[ \xrightarrow{1} 4 \xrightarrow{6} 2 \xrightarrow{3} 5 \]

So \( \sigma = c_1 \circ c_2 = c_2 \circ c_1 \).

Lemma. 1. \( \forall \sigma \in S_n, \sigma(\text{Fix}(\sigma)) = \text{Fix}(\sigma) \)

2. \( \forall \sigma_1, \sigma_2 \in S_n, \text{Fix}(\sigma_1 \cup \text{Fix}(\sigma_2)) = \{1, 2, \ldots, n\} \)

\( \Rightarrow \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \).

Proof. 1. \( x \in \text{Fix}(\sigma) \iff \sigma(x) = x \)

\( \iff \sigma(\sigma(x)) = \sigma(x) \)

\( \iff \sigma(x) \in \text{Fix}(\sigma) \).

2. \( \sigma_2(\text{Fix}(\sigma_1)) \neq \text{Fix}(\sigma_1) \) If not

\( \exists x \in \text{Fix}(\sigma_1) \) and \( \sigma_2(x) \notin \text{Fix}(\sigma_1) \).

So \( \sigma_2(x) \neq x \Rightarrow x \notin \text{Fix}(\sigma_2) \Rightarrow \sigma_2(x) \notin \text{Fix}(\sigma_2) \)

\( \Rightarrow \sigma_2(x) \in \text{Fix}(\sigma_1) \)

\( \forall x, x \in \text{Fix}(\sigma_1) \cup \text{Fix}(\sigma_2) \) WLOG let's assume

that \( x \in \text{Fix}(\sigma_1) \Rightarrow \sigma_2(x) \in \text{Fix}(\sigma_1) \)

\( \Rightarrow (\sigma_2 \circ \sigma_1)(x) = \sigma_2(x) = \sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x) \).

Definition. Two cycles \( c_1 \) and \( c_2 \) in \( S_n \) are called disjoint

\( \text{if } \text{Fix}(c_1) \cup \text{Fix}(c_2) = \{1, 2, \ldots, n\} \).

Two cycles \( c_1 \) and \( c_2 \) are denoted by

\( (i_1, i_2, \ldots, i_k) \)

Remark. 1. \( (i_1, \ldots, i_k) \) and \( (j_1, \ldots, j_l) \) (for \( k, l \geq 2 \))

are disjoint \( \iff i_s \neq j_t \) for any \( s \) and \( t \).
Proposition. Any \( \sigma \in S_n \) can be uniquely written as a product of disjoint cycles. (up to rearrangement)

\[ (i_1, i_2, \ldots, i_k) = (i_k, i_{k-1}, \ldots, i_1). \]

Proof. We have already proved the existence. Let's say a few words on the uniqueness; suppose \( c_i \)'s are disjoint cycles. Then the cycles in the Schreier graph of \( c^i \cdot c^j \) are exactly \( c_i \)'s.

\( \forall x \in \text{Fix}(c^i_j) \) except possibly for one value of \( i \).

\( \overset{\text{by induction}}{\Rightarrow} C_i \cdot C_j \subseteq \bigcap_{i \neq j} \text{Fix} c_i \)

\( = (a_1 a_2 a_3 \cdots a_n) \) if \( a_i \neq a_j \) for any \( i \neq j \).

Conclusion. Any permutation is product of 2-cycles. 2-cycles are also called transposition.

Conclusion. A \( k \)-cycle can be written as a product of \( k-1 \) transpositions.

Proposition. Suppose \( c_i \)'s are disjoint \( k_i \)-cycles. Then \( \sigma(c_1 \cdots c_k) = \text{lcm}(k_1, \ldots, k_2) \).

Proof. Let \( \sigma = c_1 \cdots c_k \). We have already proved that the cycles in the Schreier graph of \( \langle \sigma \rangle \)

are the same as \( c_i \)'s; and their length divides \( o(\sigma) \Rightarrow k \cdot o(\sigma) \).
\[ \Rightarrow \text{l.c.m}(k_1, \ldots, k_d) \mid o(\sigma_1) \quad \text{III} \]

Now let \( t = \text{l.c.m}(k_1, \ldots, k_d) \). Since \( c_i \circ c_j = c_j \circ c_i \)
we have \( \sigma^t = c_1^t \circ \cdots \circ c_d^t \Rightarrow o(t) = \text{id} \).

Now let \( k_i \mid t \Rightarrow c_i^t = \text{id} \).

\[ \Rightarrow \sigma(\sigma_1) \mid t \quad \text{II} \]

Hence III and II \( \Rightarrow o(\sigma) = \text{l.c.m}(k_1, \ldots, k_d) \quad \blacksquare \)

**Proposition (Two Important Equalities)**

1. \((a_1, \ldots, a_n)^{-1} = (a_n, \ldots, a_1)\)
2. \(\tau(a_1, \ldots, a_n)^{-1} = (\tau(a_1), \ldots, \tau(a_n))\)

**Proof (2)**

\[
\begin{align*}
(a_1, \ldots, a_n)^{-1} &= (a_n, \ldots, a_1) \\
\tau(a_1, \ldots, a_n)^{-1} &= \tau(a_n, \ldots, a_1) \\
\end{align*}
\]

\[
\begin{cases}
\tau(a_{i+1}) & \text{if } i \neq n \\
\tau(a_i) & \text{if } i = n \\
\end{cases}
\]

If \( x \notin \{\tau(a_1), \ldots, \tau(a_n)\} \Rightarrow \\
\tau^d(x) \notin \{a_1, \ldots, a_d\} \Rightarrow \\
(a_1, \ldots, a_n)(\tau^d(x)) = \tau^d(x) \Rightarrow \\
(\tau(a_1, \ldots, a_n)(\tau^d(x)) = x \quad \blacksquare \)