# Lecture 18: permutations.

Wednesday, November 12, 2014
12:12 PM

In the previous lecture we observed that

$\langle a \rangle \curvearrowright X$ and $|\langle a \rangle| < \infty \Rightarrow X$ can be viewed as

disjoint union of cycles s.t.

① Each cycle consists of <u>one orbit</u>.

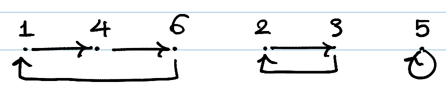
② length of each cycle divides $|\langle a \rangle| = o(a)$.

③ $a$ acts by rotating <u>1</u>-step on each cycle.

For instance, if $o(a) = p$, then all the cycles are either of size 1 (fixed points) or of size $p$.

One important example is the action of $\langle \sigma \rangle$ on $\{1, 2, \dots, n\}$ where $\sigma \in S_n$. Let's consider the following element of $S_6$:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 6 & 5 & 1 \end{array}$$



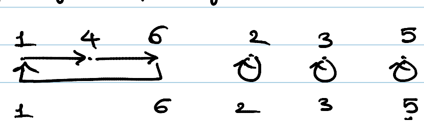

cycle of length 3     cycle of length 2     cycle of length 1

So we can understand <u>$\sigma$</u> by looking at these <u>cycles</u>.

- $\forall \sigma \in S_n$, let $\mathrm{Fix}(\sigma) := \{i \mid \sigma(i) = i\}$.
  These give us cycles of length 1.

- We say $\sigma$ is a cycle of length <u>$k$</u> if its Schreier graph consists of one cycle of length $k$ and bunch of cycles of length 1.

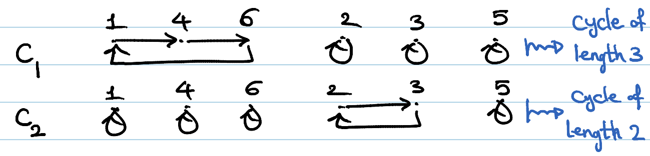
<u>Exp.</u> $c_1$



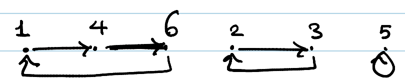

$\rightsquigarrow$ Cycle of length 3

Cycle of

Exp. $c_1$, $c_2$




What are $c_1 \circ c_2$ and $c_2 \circ c_1$?




So $\sigma = c_1 \circ c_2 = c_2 \circ c_1$.

<u>Lemma</u>. ① $\forall \sigma \in S_n$, $\sigma(\mathrm{Fix}(\sigma)) = \mathrm{Fix}\,\sigma$

② $\forall \sigma_1, \sigma_2 \in S_n$, $\mathrm{Fix}\,\sigma_1 \cup \mathrm{Fix}\,\sigma_2 = \{1, 2, \cdots, n\}$

$\Rightarrow \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$.

<u>Pf</u>. ① $x \in \mathrm{Fix}\,\sigma \Leftrightarrow \sigma(x) = x$

$\Leftrightarrow \sigma(\sigma(x)) = \sigma(x)$

$\Leftrightarrow \sigma(x) \in \mathrm{Fix}\,\sigma$.

② $\sigma_2(\mathrm{Fix}\,\sigma_1) \overset{?}{=} \mathrm{Fix}\,\sigma_1$. If not

$\exists$ $x \in \mathrm{Fix}\,\sigma_1$ and $\sigma_2(x) \notin \mathrm{Fix}\,\sigma_1$.

So $\sigma_2(x) \neq x \Rightarrow x \notin \mathrm{Fix}\,\sigma_2 \Rightarrow \sigma_2(x) \notin \mathrm{Fix}\,\sigma_2$

$\Rightarrow \sigma_2(x) \in \mathrm{Fix}\,\sigma_1$ ※

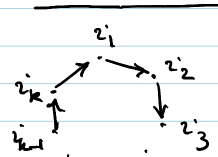
- $\forall x$, $x \in \mathrm{Fix}\,\sigma_1 \cup \mathrm{Fix}\,\sigma_2$. W.L.O.G let's assume

that $x \in \mathrm{Fix}\,\sigma_1$. $\Rightarrow \sigma_2(x) \in \mathrm{Fix}\,\sigma_1$

$\Rightarrow (\sigma_2 \circ \sigma_1)(x) = \sigma_2(x) = \sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x)$. ∎

<u>Def</u>. • Two cycles $c_1$ and $c_2$ in $S_n$ are called disjoint

if $\mathrm{Fix}\,c_1 \cup \mathrm{Fix}\,c_2 = \{1, 2, \cdots, n\}$.

- 

is denoted by

$(i_1, i_2, \cdots, i_k)$

<u>Remark</u>. ① $(i_1, \cdots, i_k)$ and $(j_1, \cdots, j_\ell)$ (for $k, \ell \geq 2$)

are disjoint $\Leftrightarrow$ $i_s \neq j_t$ for any $s$ and $t$.

② $(i_1, i_2, \ldots, i_k) = (i_k, i_1, \ldots, i_{k-1})$.

<u>Proposition</u>. Any $(1) \neq \sigma \in S_n$ can be uniquely written as product of disjoint cycles. (up to rearrangement).

<u>Pf</u>. We have already proved the existence. Let's say a few words on the uniqueness: Suppose $c_i$'s are disjoint cycles. Then the cycles in the Schreier graph of $\sigma = c_1 \circ \cdots \circ c_\ell$ are exactly $c_i$'s.:

$\forall x$ is in <u>$\mathrm{Fix}(c_j)$</u> except possibly for one value of $j$. $\rightsquigarrow$ $\sigma(x) = c_{j_0}(x) \in \bigcap_{j \neq j_0} \mathrm{Fix}\, c_j$

$\Rightarrow$ by induction,

$$\sigma^{(i)}(x) = c_{j_0}^{(i)}(x). \qquad \blacksquare$$

<u>Exp</u>. $(1\ 3)(2\ 3) = (1\ 3\ 2)$

$$(a_1\ a_2)(a_2\ a_3)(a_3\ a_4) \cdots (a_{n-1}\ a_n)$$
$$= (a_1\ a_2\ a_3 \cdots a_n) \quad \text{if } a_i \neq a_j \text{ for any } i \neq j.$$

<u>Cor</u>. Any permutation is product of <u>2-cycles</u>.

2-cycles are also called <u>transposition</u>.

<u>Cor</u>. A $k$-cycle can be written as a product of <u>$k-1$</u> transpositions.

<u>Proposition</u>. Suppose $c_i$'s are disjoint $k_i$-cycles.

Then $o(c_1 \circ \cdots \circ c_\ell) = \mathrm{lcm}(k_1, \ldots, k_\ell)$.

<u>Pf</u>. Let $\sigma = c_1 \circ \cdots \circ c_\ell$. We have already proved that the cycles in the Schreier graph of $\langle \sigma \rangle$ are the same as $c_i$'s; and their length divides $o(\sigma)$. $\Rightarrow k_i \mid o(\sigma)$

$$\Rightarrow \ \text{l.c.m}(k_1, \dots, k_\ell) \mid o(\sigma). \quad \text{I}$$

Now let $t = \text{l.c.m}(k_1, \dots, k_\ell)$. Since $c_i \circ c_j = c_j \circ c_i$

we have $\left.\begin{array}{l} \sigma^t = c_1^t \circ \dots \circ c_\ell^t \\ k_i \mid t \Rightarrow c_i^t = id. \end{array}\right\} \Rightarrow \sigma^t = id.$

$$\Rightarrow o(\sigma) \mid t. \quad \text{II}$$

Hence I and II $\Rightarrow o(\sigma) = \text{lcm}(k_1, \dots, k_\ell)$. ■

<u>Proposition</u> (Two Important Equality)

(i) $(a_1, \dots, a_n)^{-1} = (a_n, \dots, a_1)$

(ii) $\tau\,(a_1, \dots, a_n)\,\tau^{-1} = (\tau(a_1), \dots, \tau(a_n))$

<u>Pf</u> (i) 


(ii) $(\tau\,(a_1, \dots, a_n)\,\tau^{-1})(\tau(a_i))$

$= (\tau\,(a_1, \dots, a_n))\,(a_i)$

$= \begin{cases} \tau(a_{i+1}) & \text{if } i \neq n \\ \tau(a_1) & \text{if } i = n. \end{cases}$

If $x \notin \{\tau(a_1), \dots, \tau(a_n)\} \Rightarrow$

$\tau^{-1}(x) \notin \{a_1, \dots, a_n\} \Rightarrow$

$(a_1, \dots, a_n)(\tau^{-1}(x)) = \tau^{-1}(x) \Rightarrow$

$(\tau\,(a_1, \dots, a_n)\,\tau^{-1})(x) = x$. ■