Lecture 21: group homomorphism.

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10:24 AM

Def. $\phi : G_1 \longrightarrow G_2$ is called a group homomorphism

if $\forall\, a, b \in G_1$, $\phi(ab) = \phi(a)\phi(b)$.

Examples that we have seen

- $\pi_n : \mathbb{Z} \longrightarrow \mathbb{Z}_n$, $\pi_n(a) := [a]_n$
- $\pi_{n,m} : \mathbb{Z}_n \longrightarrow \mathbb{Z}_m$, if $m \mid n$, $\pi_{n,m}([a]_n) = [a]_m$.
- $\mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ if $\gcd(m,n)$

  $[a]_{mn} \longmapsto ([a]_m, [a]_n)$ is a group homomorphism which is also a bijection.
- $\mathbb{Z}_{o(g)} \longrightarrow \langle g \rangle$ is a bijection which is

  $[a]_{o(g)} \longmapsto g^a$ also a group homomorphism.
- $G \curvearrowright X \Rightarrow \rho : G \longrightarrow S_X$

  $$\rho(g)(x) := g \cdot x$$

Def. A group homomorphism θ is called a monomorphism if it is 1-1. θ is called an epimorphism if it is <u>onto</u>. θ is called an <u>isomorphism</u> if θ is a bijection.

- π_n is an epimorphism
- $\pi_{n,m}$ is an epimorphism. It is an isomorphism if and only if $n=m$.
- Any cyclic group of order n is isomorphic to $\mathbb{Z}_n$.
- $\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$ if $\gcd(m,n)=1$.
- $\mathrm{sgn}: S_n \longrightarrow \{\pm 1\}$ is an epimorphism if $n \geq 2$.

<u>Def</u>. Let $\phi: G \to H$ be a group homomorphism.

$$\text{Image of } \phi = \mathrm{Im}(\phi) = \{\phi(g) \mid g \in G\} \subseteq H$$

And kernel of $\phi = \ker(\phi) = \{g \in G \mid \phi(g) = e\} \subseteq G$.