Lecture 21: group homomorphism.

Def. \( \phi : G_1 \rightarrow G_2 \) is called a group homomorphism if \( \forall a, b \in G_1, \ \phi(ab) = \phi(a) \phi(b) \).

Examples that we have seen

1. \( \pi_n : \mathbb{Z} \rightarrow \mathbb{Z}_n, \ \pi_n(a) := [a]_n \)

2. \( \pi_{n,m} : \mathbb{Z}_n \rightarrow \mathbb{Z}_m, \text{ if } m \mid n, \ \pi_{n,m}([a]_n) = [a]_m \)

3. \( \mathbb{Z}_{mn} \rightarrow \mathbb{Z}_m \times \mathbb{Z}_n \text{ if } \gcd(m,n) \)

\([a]_{mn} \mapsto ([a]_m, [a]_n)\) is a group homomorphism which is also a bijection.

4. \( \mathbb{Z}_{o(g)} \rightarrow \langle g \rangle \) is a bijection which is \( [a]_{o(g)} \mapsto g^a \) also a group homomorphism.

5. \( G \cap X \Rightarrow \rho : G \rightarrow S_X \)

\( \rho(g)(x) := g \cdot x \)

Def. A group homomorphism \( \Theta \) is called a monomorphism if it is 1-1. \( \Theta \) is called an epimorphism if it
is onto. \( \Theta \) is called an isomorphism if \( \Theta \) is a bijection.

- \( \pi_n \) is an epimorphism

- \( \pi_{n,m} \) is an epimorphism. It is an isomorphism if and only if \( n = m \).

- Any cyclic group of order \( n \) is isomorphic to \( \mathbb{Z}_n \).

- \( \mathbb{Z}_{mn} \) is isomorphic to \( \mathbb{Z}_m \times \mathbb{Z}_n \) if \( \text{gcd}(m,n) = 1 \).

- \( \text{sgn}: S_n \to \{\pm 1\} \) is an epimorphism if \( n \geq 2 \).

**Def.** Let \( \phi: G \to H \) be a group homomorphism.

\[
\text{Image of } \phi = \text{Im}(\phi) = \{ \phi(g) \mid g \in G \} \subseteq H
\]

And \( \text{kernel of } \phi = \text{ker}(\phi) = \{ g \in G \mid \phi(g) = e \} \subseteq G \).