Lecture 28: examples

Wednesday, December 10, 2014
12:01 AM

Exp. $|G| = 4 \Rightarrow$ either $G \simeq \mathbb{Z}_4$ or $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Pf. $\forall g \in G$, $o(g) \mid 4 \Rightarrow o(g) = 1$ or 2 or 4

If $\exists g \in G$, $o(g) = 4 \Rightarrow G$ is cyclic $\Rightarrow G \simeq \mathbb{Z}_4$.

If not, $\forall g \in G$, $g^2 = e \Rightarrow G$ is abelian.

$b \notin \{e, a\} \Rightarrow G = \{e, a, b, ab\}$

$\Rightarrow$

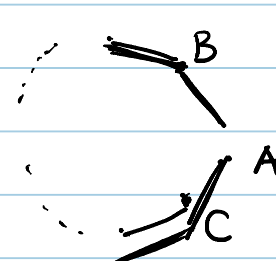
$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow G$$
$$([i]_2, [j]_2) \longmapsto a^i b^j$$
is an isomorphism.

Exp. • Elements of the n^{th} dihedral group D_n.

- The set of rotations is a normal subgroup.
- The table of multiplication.

Solution. Suppose σ is a symmetry of a regular n-gon.



So $\sigma(A)$ is a vertex. Hence, for some i,

$$b^{-i}\sigma(A) = A,$$

where b is the $\frac{2\pi}{n}$ rotation about the origin

$b^{-i}\sigma$ is a symmetry and sends A to $A \Longrightarrow$

either $b^{-i}\sigma(B) = B$ or $b^{-i}\sigma(B) = C$.

$\Longrightarrow$ either $\begin{cases} b^{-i}\sigma(A) = A \\ b^{-i}\sigma(B) = \end{cases}$

<u>Exp.</u> Find $Z(D_n)$.

<u>Exp.</u> $|G| = 2p \Longrightarrow$ either $G \simeq \mathbb{Z}_{2p}$ or $G \simeq D_p$.

<u>Solution.</u> By Cauchy's theorem,

$$\exists\, \alpha, \beta \in G, \quad o(\alpha) = 2 \text{ and } o(\beta) = p.$$

$\Longrightarrow \alpha \notin \langle\beta\rangle \Longrightarrow G = \langle\beta\rangle \cup \alpha\langle\beta\rangle.$

and $\langle\beta\rangle \triangleleft G$.

$\Longrightarrow \alpha\beta\alpha^{-1} = \beta^{i}$

$\Longrightarrow \alpha(\alpha\beta\alpha^{-1})\alpha^{-1} = \alpha\beta^{i}\alpha^{-1} = \beta^{i^2}$

$\Longrightarrow \beta = \beta^{i^2} \Longrightarrow i^2 \equiv 1 \pmod{p}$

$\Longrightarrow \underbrace{i \overset{p}{\equiv} 1}$ or $\underbrace{i \overset{p}{\equiv} -1}$.

$$\begin{array}{ll} \Downarrow & \Downarrow \\ \alpha\beta=\beta\alpha & \alpha\beta\alpha^{-1}=\beta^{-1} \\ \Downarrow & \Downarrow \\ o(\alpha\beta)=2p & a \mapsto \alpha \\ \Downarrow & b \mapsto \beta \\ G \text{ is cyclic} & \text{extends to an} \\ \Downarrow & \text{isomorphism.} \\ G \simeq \mathbb{Z}_{2p} & \text{(Looking at the table)} \end{array}$$

$\square$

Exp. $Z(S_n) = \{e\}$, $\quad n \geq 3$.