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1. (10 points) Show that $\mathbb{Z}_{12} \times \mathbb{Z}_9$ is not cyclic.

2. (10 points) Let $G = \langle a \rangle$ be a finite group of size $n$. Show that
   $$|\{g \in G\mid o(g) = n\}| = \phi(n).$$

3. Let $H$ be a subgroup of $G = \langle a \rangle$.
   (a) (5 points) Show that $I_H := \{m \in \mathbb{Z} \mid a^m \in H\}$ is a subgroup of $\mathbb{Z}$.
   (b) (5 points) Show that $H$ is cyclic.

4. (10 points) Let $G$ be a finite group, $p$ be a prime, and $X$ be a finite set. Suppose $G$ acts from left on $X$, and $|G| = p$. Show that for any $x \in X$ either $x$ is a fixed point, i.e. for any $g \in G$ we have $g \cdot x = x$, or the size $|O(x)|$ of the orbit of $x$ is divisible by $p$.
   
   Hint: Think about the connection between $O(x)$ and the stabilizer subgroup $G_x$. And use Lagrange!

5. Let $\sigma = \left(\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 8 & 1 & 2 & 5 & 3 & 4 & 7
\end{array}\right)$.
   (a) (3 points) Write $\sigma$ as a product of disjoint cycles.
   (b) (5 points) Find $o(\sigma)$. Justify your answer.
   (c) (2 points) Are there a 3-cycle $c$ and a 5-cycle $c'$ such that $o(cc') = 8$ (not necessarily disjoint)? Justify your answer. (This part has nothing to do with the first two parts!)

6. (10 points) (EXTRA CREDIT) Can the following arrangement happen in the 15-puzzle? Justify your answer.

   \[
   \begin{array}{cccc}
   2 & 1 & 4 & 3 \\
   6 & 5 & 8 & 9 \\
   7 & 14 & 10 & 11 \\
   12 & 13 & 15 \\
   \end{array}
   \]

   (Hint:

   1. Think about permutations in $S_{16}$.
   2. What is a single slide as a permutation?
   3. What can you say about the number of slides to get to this arrangement?)

   Good Luck!