QUIZ 1, VERSION A, MATH100B, WINTER 2021

1. Answer the following questions and briefly justify your answers.
   (a) (1 point) True or false. Every integral domain can be embedded into a field.
   (b) (2 point) Find \(|(\mathbb{Z}[x])^\times|\).
   (c) (3 points) True or false. There is an integral domain \(D\) such that
   \[
   \underbrace{1_D + \cdots + 1_D}_{9 \text{ times}} = 0 \quad \text{and} \quad 1_D + 1_D + 1_D \neq 0.
   \]
   (d) (4 points) Find \(|(\mathbb{Z}_9 \times \mathbb{Z}_5)^\times|\).

2. (5 points) Prove that \(\mathbb{Q}[x]/(x^2 - 3) \simeq \mathbb{Q}[\sqrt{3}]\) where \(\mathbb{Q}[\sqrt{3}]\) is the smallest subring of \(\mathbb{C}\) that contains \(\mathbb{Q}\) and \(\sqrt{3}\).

3. (5 points) Suppose \(p\) is a prime number and \(f(x) \in \mathbb{Z}_p[x]\) is a polynomial of degree 3. Use the long
division for polynomials to prove that \(|\mathbb{Z}_p[x]/\langle f(x) \rangle| = p^3\).

4. Suppose \(m\) and \(n\) are positive integers and \(\gcd(m, n) = 1\). Let \(e : \mathbb{Z} \rightarrow \mathbb{Z}_n \times \mathbb{Z}_m, \ e(k) := k([1]_n, [1]_m)\).
   You can use without proof that \(e\) is a ring homomorphism.
   (a) (3 points) Find the kernel of \(e\).
   (b) (4 points) Prove that \(e\) is surjective.
   (c) (3 points) Prove that \(\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}_n \times \mathbb{Z}_m\).