QUIZ 1, VERSION B, MATH100B, WINTER 2021

1. Answer the following questions and briefly justify your answers.
   (a) (1 points) True or false. Every integral domain is a field.
   (b) (2 point) True or false. A field has exactly two ideals.
   (c) (3 points) Find the characteristic of $\mathbb{Z}_3 \times \mathbb{Z}_7$.
   (d) (4 points) Find $|(\mathbb{Z}_{25} \times \mathbb{Z}_7)^{\times}|$.

2. Let's recall that $\mathbb{Q}[i] = \{a + bi | \ a, b \in \mathbb{Q}\}$ is a subring of $\mathbb{C}$.
   (a) (4 points) Prove that $\mathbb{Q}[x]/\langle x^2 + 1\rangle \simeq \mathbb{Q}[i]$
   (b) (2 points) Prove that $\mathbb{Q}[x]/\langle x^2 + 1\rangle$ is a field.

3. (4 points) Suppose p is prime. Prove that $x^{p^2} - x + 1$ has no zero in $\mathbb{Z}_p$.

4. (4 points) Suppose $\alpha \in \mathbb{C}$ is a zero of a polynomial $p(x) \in \mathbb{Q}[x]$ of degree 3. Use the long division for polynomials to prove that $\mathbb{Q}[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 | \ a_0, a_1, a_2 \in \mathbb{Q}\}$.

5. Let's recall that $\mathbb{Z}[i] := \{a + bi | \ a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$.
   (a) (2 points) Suppose p is a prime and there is a ring homomorphism $f : \mathbb{Z}[i] \to \mathbb{Z}_p$ such that $f(1) = 1$. Prove that there is $x \in \mathbb{Z}_p$ such that $x^2 = -1$.
   (b) (4 points) Find a surjective ring homomorphism $f : \mathbb{Z}[i] \to \mathbb{Z}_{13}$ such that $3-2i \in \ker f$. (Notice that $8^2 + 1$ is a multiple of 13.)