1. Answer the following questions and briefly justify your answers.
   (a) (1 point) True or false. Every integral domain is a field.
   (b) (2 points) True or false. A field has exactly two ideals.
   (c) (3 points) Find the characteristic of \( \mathbb{Z}_3 \times \mathbb{Z}_7 \).
   (d) (4 points) Find \(|(\mathbb{Z}_{25} \times \mathbb{Z}_7)^x|\).

2. Let’s recall that \( \mathbb{Q}[i] = \{a + bi| a, b \in \mathbb{Q}\} \) is a subring of \( \mathbb{C} \).
   (a) (4 points) Prove that \( \mathbb{Q}[x]/(x^2 + 1) \simeq \mathbb{Q}[i] \)
   (b) (2 points) Prove that \( \mathbb{Q}[x]/(x^2 + 1) \) is a field.

3. (4 points) Suppose \( p \) is prime. Prove that \( x^{p^2} - x + 1 \) has no zero in \( \mathbb{Z}_p \).

4. (4 points) Suppose \( \alpha \in \mathbb{C} \) is a zero of a polynomial \( p(x) \in \mathbb{Q}[x] \) of degree 3. Use the long division for polynomials to prove that \( \mathbb{Q}[\alpha] = \{a_0 + a_1 \alpha + a_2 \alpha^2| a_0, a_1, a_2 \in \mathbb{Q}\} \).

5. Let’s recall that \( \mathbb{Z}[i] := \{a + bi| a, b \in \mathbb{Z}\} \) is a subring of \( \mathbb{C} \).
   (a) (2 points) Suppose \( p \) is a prime and there is a ring homomorphism \( f : \mathbb{Z}[i] \rightarrow \mathbb{Z}_p \) such that \( f(1) = 1 \). Prove that there is \( x \in \mathbb{Z}_p \) such that \( x^2 = -1 \).
   (b) (4 points) Find a surjective ring homomorphism \( f : \mathbb{Z}[i] \rightarrow \mathbb{Z}_{13} \) such that \( 3 - 2i \in \ker f \). (Notice that \( 8^2 + 1 \) is a multiple of 13.)