QUIZ 2, VERSION A, MATH100B, WINTER 2021

1. Answer the following questions and briefly justify your answers.
   (a) (2 point) Find all primes \( p \) such that \( x - 1 \) is a factor of \( x^5 - 2x^4 + 3x^3 + 5x^2 + 6 \) in \( \mathbb{Z}_p \).
   (b) (3 points) True or false, \( \mathbb{Z}[x] \) is a PID.

2. (5 points) Determine whether \( f(x) := x^5 - 2x^4 + 5x^3 - x + 1 \) has a zero in \( \mathbb{Q} \). Justify your answer.

3. Recall that \( \mathbb{Z}[i] = \{ a + bi \mid a, b \in \mathbb{Z} \} \).
   (a) (4 points) Prove that \( 5 + 2i \) is irreducible in \( \mathbb{Z}[i] \).
      (Hint: Think about \( N(a + bi) = |a + bi|^2 = a^2 + b^2 \).)
   (b) (4 points) Prove that \( \mathbb{Z}[i]/\langle 5 + 2i \rangle \) is a field.
   (c) (2 points) Prove that the characteristic of \( \mathbb{Z}[i]/\langle 5 + 2i \rangle \) is 29.

4. Suppose \( E \) is a field extension of \( \mathbb{Z}_3 \), and \( \alpha \in E \) is a zero of \( x^3 - x + 2 \).
   (a) (6 points) Prove that \( \mathbb{Z}_3[\alpha] \) is a field of order 27.
   (b) (2 points) Prove that \( \alpha^{26} = 1 \). (Hint: Think about \( (\mathbb{Z}_3[\alpha])^\times \)).
   (c) (2 points) Prove that \( x^3 - x + 2 \) divides \( x^{26} - 1 \).