QUIZ 3, VERSION B, MATH100B, WINTER 2021

1. (5 points) Suppose p is prime. Prove that $x^{p-1} + x^{p-2} + \cdots + 1 \in \mathbb{Q}[x]$ is irreducible.

2. (5 points) Suppose every ideal of a unital commutative ring A is finitely generated. Prove that A is Noetherian.

3. Suppose A is a subring of B, B is a unital commutative ring, $1_B \in A$, and I is an ideal of B.
   (a) (3 points) Prove that $f : A \to B/I,\ f(a) := a + I$ is a ring homomorphism and $\ker f = I \cap A$.

   (b) (5 points) Prove that if I is a prime ideal of B, then $I \cap A$ is a prime ideal of A.

   (c) (2 points) Provide an example where I is a maximal ideal of B, but $I \cap A$ is not a maximal ideal of A.

4. Suppose p is prime and $f(x) := (x^p - x + 1)^2 + p$.
   (a) (5 points) Suppose $f(x) = q(x)h(x)$ for some monic non-constant polynomials $q, h \in \mathbb{Z}[x]$. Prove that there are polynomial $q_1, h_1 \in \mathbb{Z}[x]$ such that
   $$q(x) = x^p - x + 1 + p\, q_1(x), \text{ and } h(x) = x^p - x + 1 + p\, h_1(x).$$
   (You are allowed to use a relevant result from HW assignment after you carefully state it.)

   (b) (3 points) Suppose q_1 and h_1 are as in the previous part. Prove that
   $$(x^p - x + 1)(q_1 + h_1) \equiv 1 \pmod{p}$$
   and discuss why this is a contradiction.

   (c) (2 points) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.