QUIZ 4, VERSION B, MATH100B, WINTER 2021

1. (7 points) Prove that $\mathbb{Q}[\zeta_n, \sqrt[n]{2}]$ is a splitting field of $x^n - 2$ over $\mathbb{Q}$ where $\zeta_n := e^{2\pi i/n} \in \mathbb{C}$.

2. (5 points) Suppose F is a field and $f(x) \in F[x]$ is irreducible. Prove that there is a field extension E of F which contains a zero of f.

3. (3 points) Find all the primes p such that $x^{12} - 1$ has multiple zeros in its splitting field over $\mathbb{Z}_p$.

4. Suppose F is a field $f \in F[x]$, and $\alpha \in E$ is a zero of f. Let E be a splitting field of f over F and L be a field extension of E. Suppose $\theta : E \to L$ is an injective ring homomorphism and $\theta(c) = c$ for every $c \in F$.
   (a) (3 points) Suppose $\alpha \in E$ is a zero of f. Prove that $\theta(\alpha)$ is also a zero of f, and deduce that θ permutes zeros of f.
   (b) (2 points) Prove that $\theta(E) = E$.

5. (10 points) Suppose p is prime and n is a positive integer. Prove that there is an irreducible polynomial of degree n in $\mathbb{Z}_p[x]$. (In this question, you are allowed to use the fact that the group of units of a finite field is cyclic and other results about finite fields that are proved in class.)