# QUIZ 2, MATH100C, SPRING 2021

For questions 3 and 4 you can use the following theorem from group theory:

*Suppose $p$ is prime and $G$ is a subgroup of $S_p$ which acts transitively on $\{1, \ldots, p\}$; that means for every $1 \le i, j \le p$ there is $\sigma \in G$ such that $\sigma(i) = j$. Then $G$ is solvable if and only if every non-trivial element of $G$ fixes at most one point; that means if $\sigma \in G$, $\sigma(i) = i$, and $\sigma(j) = j$ for distinct values $i$ and $j$, then $\sigma = \mathrm{id}$.*

1. Suppose $F$ is a field of characteristic $p > 0$. Suppose $E/F$ is a purely inseparable extension and $L/E$ is an algebraic extension. Suppose $\alpha \in L$.
   (a) (1 point) Prove that $m_{\alpha,E}$ divides $m_{\alpha,F}$ in $E[x]$.
   (b) (2 points) Prove that for some integer power $q$ of $p$, we have $m_{\alpha,E}^q \in F[x]$.
   (c) (1 point) Prove that for some integer power $q$ of $p$, $m_{\alpha,F}$ divides $m_{\alpha,E}^q$ in $E[x]$.
   (d) (4 points) Prove that if $m_{\alpha,F}$ is separable in $F[x]$, then $[F[\alpha] : F] = [E[\alpha] : E]$.

2. Suppose $F$ is a field of characteristic zero and $f \in F[x]$ is a monic irreducible polynomial. Let $E$ be a splitting field of $f$ over $F$. Suppose $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$ in $E[x]$. Let $G := \mathrm{Aut}_F(E)$ and $G_i := \mathrm{Aut}_{F[\alpha_i]}(E)$.
   (a) (3 points) Prove that there is $\sigma_i \in G$ such that $\sigma_i(\alpha_1) = \alpha_i$ for every $i$.
   (b) (3 points) Prove that $G_i = \sigma_i G_1 \sigma_i^{-1}$ for every $i$.
   (c) (3 points) Prove that if $G$ is abelian, then $E = F[\alpha_1]$.

3. (5 points) Let $F$ be a characteristic zero field. Suppose $p$ is a prime number, and $f$ is a monic irreducible polynomial of degree $p$ in $F[x]$. Let $E$ be a splitting field of $f$ over $F$. Suppose $f$ is solvable by radicals over $F$. Prove that if $\alpha$ and $\alpha'$ are two distinct zeros of $f$, then $E = F[\alpha, \alpha']$.

4. Suppose $p$ is a prime number more than 4, and $f(x) = x^p - 4x + 2 \in \mathbb{Q}[x]$.
   (a) (1 point) Prove that $f'$ has exactly 2 real zeros in $\mathbb{C}$.
   (b) (2 points) Prove that $f$ has exactly 3 real zeros in $\mathbb{C}$.
   (c) (2 points) Prove that $f$ is irreducible in $\mathbb{Q}[x]$.
   (d) (3 points) Prove that $f$ is not solvable by radicals over $\mathbb{Q}$.