QUIZ 3 VERSION B SOLUTIONS
MATH 103A, SUMMER 2021

(1) (a) True. Since $[2]_9$ generates the cyclic subgroup

$$\langle [2]_9 \rangle = \{[2]_9, [4]_9, [8]_9, [7]_9[5]_9, [1]_9\}$$

of order 6. We conclude that $o([2]_9) = |\langle [2]_9 \rangle| = 6$.

(b) False. Suppose $f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is an isomorphism. Then $f(1) = 0$. Let $f(-1) = k$. As f is a group homomorphism,

$$0 = f(1) = f((-1) \cdot (-1)) = f(-1) + f(-1) = 2k.$$

Hence $k = 0$ and f is not injective. Hence such an isomorphism does not exist.

(c) False. Consider the elements $\sigma_1 = (1, 2, 3, 4)$ and $\sigma_2 = (5, 6, 7, 8, 9)$ in S_9. Both σ_1 and σ_2 are cycles with orders 4 and 5 respectively. Hence

$$o(\sigma_1\sigma_2) = \text{lcm}(4, 5) = 20.$$

So $\sigma_1\sigma_2$ is an element of order 20 in S_9.

(d) True. Let $\tau_1 = (1, 2)$, $\tau_2 = (2, 3)$. By the linking lemma, $\tau_1\tau_2 = (1, 2)(2, 3) = (1, 2, 3)$ and $o(\tau_1\tau_2) = 3$.

(2) (a) Since G is cyclic of order 70, $x^{70} = y^{70} = e_G$. Hence in $G \times G$,

$$(x, y)^{70} = (x^{70}, y^{70}) = (e_G, e_G).$$

(b) Suppose $G \times G$ is cyclic and has generator (x, y). Then $o(x, y) = |G \times G|$. From part (a), $o(x, y)||G|$ hence $o(x, y) \leq |G|$. We now note,

$$o(x, y) \leq |G| < |G \times G|$$

a contradiction. Hence no such generator exists and $G \times G$ is not cyclic.

(c) Using the formula

$$o(g^k) = \frac{o(g)}{\gcd(k, o(g))}$$

from the lecture, we obtain

$$14 = o(g^k) = \frac{o(g)}{\gcd(o(g), k)} = \frac{70}{\gcd(k, 70)}.$$

Hence $\gcd(k, 70) = 5$.

(d) Since G is cyclic any element of G is of the form g^k for some k satisfying $0 \leq k < 70$. By the previous part $o(g^k) = 14$ implies $\gcd(k, 70) = 5$. Since

$$\gcd(k, 70) = 5 \iff \gcd(l, 14) = 1$$

where $5l = k$ and $1 \leq 5l \leq 70$. There are only $\phi(14) = 6$ possible values of l, hence there are 6 elements of order 14.

(e) We recall from lecture that any finite cyclic group has one subgroup for each divisor of the order of the group. Since the divisors of 70 are $\{1, 2, 5, 7, 10, 14, 35, 70\}$, G has 8 subgroups.

(3) (a) By repeatedly applying σ, we get the cycles $1 \to 9 \to 7 \to 3 \to 1$, $2 \to 2$, $4 \to 10 \to 6 \to 8 \to 5 \to 4$. Hence the cyclic decomposition is given by

$$\sigma = (1, 9, 7, 3)(2)(4, 10, 6, 8, 5) = (1, 9, 7, 3)(4, 10, 6, 8, 5).$$

(b) Note that the disjoint cycles that appear in the cyclic decomposition have orders 4 and 5, hence

$$|\langle\sigma\rangle| = \text{lcm}(4, 5) = 20.$$

(c) Let $\sigma_1 = (1, 9, 7, 3)$ and $\sigma_2 = (4, 10, 6, 8, 5)$. Since σ_1 and σ_2 are disjoint cycles, they commute. As $o(\sigma_1) = 4$ and $o(\sigma_2) = 5$,

$$\sigma^{59} = \sigma_1^{59}\sigma_2^{59} = \sigma_1^{-1}\sigma_2^{-1} = (3, 7, 9, 1)(5, 8, 6, 10, 4)$$

(d) Since σ_1 is an odd cycle and σ_2 is an even cycle,

$$\text{sgn}(\sigma) = \text{sgn}(\sigma_1)\text{sgn}(\sigma_2) = (-1) \cdot (1) = -1.$$

So σ is an odd cycle.