Transpositions

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In the previous video we have seen the following properties of symmetric group:

Cycle decomposition. Every non-identity element of S_n can be written as a product disjoint cycles and this decomposition is unique up to reordering the cycles.

The linking relation Suppose a_i's are pairwise distinct elements of $[1..n]$. Then

$$(a_1, \dots, a_m)(a_m, a_{m+1}, \dots, a_n) = (a_1, \dots, a_n).$$

- A 2-cycle (a_1, a_2) is called a transposition.

Lemma. Every cycle can be written as a product of transposition.

Pf. By induction on m, we prove that if a_i's are pairwise distinct, then

$$(a_1, \dots, a_m) = (a_1, a_2)(a_2, a_3) \cdots (a_{m-1}, a_m).$$

The base case of $m=2$ is clear. (the linking relation)

Induction step. $(a_1, \dots, a_m, a_{m+1}) = (a_1, \dots, a_m)(a_m, a_{m+1})$

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By the induction hypothesis $(a_1, \dots, a_m) = (a_1, a_2) \cdots (a_{m-1}, a_m)$,

and so $(a_1, a_2, \dots, a_{m+1}) = (a_1, \dots, a_m)(a_m, a_{m+1})$

$$= (a_1, a_2) \cdots (a_{m-1}, a_m)(a_m, a_{m+1}).$$

This completes the proof. ∎

Proposition. Every permutation can be written as a product of transpositions.

Pf. Suppose $\sigma \in S_n$. Then there are cycles $\sigma_1, \dots, \sigma_k$ such that $\sigma = \sigma_1 \cdots \sigma_k$ (by the cycle decomposition). By the previous lemma, each σ_i can be written as a product of transpositions. Hence σ can be written as a product of transpositions. ∎

Notice that a permutation can be written as a product of transpositions in many ways. In order to give interesting examples, let's recall that for every $\sigma \in S_n$ $\sigma(a_1, \dots, a_m)\sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_m))$. Let's also point out that $(a_1, a_2)(a_1, a_2) = id$, and so if τ

is a transposition, then $\tau^2 = \mathrm{id}.$; hence $\tau^{-1} = \tau$.

Using these relations we obtain that

$$\underbrace{(1,2)}_{\tau}(1,3)\underbrace{(1,2)}_{\tau^{-1}} = (\tau(1), \tau(3)) = (2,3).$$

This is an example of writing a permutation as a product of transpositions in different ways. An amazing fact, however, is that if

$$\tau_1 \tau_2 \cdots \tau_n = \sigma_1 \sigma_2 \cdots \sigma_m$$

and τ_i's and σ_j's are transpositions, then $m \overset{2}{\equiv} n$; this means either both m and n are odd or both of them are even. (We say m and n have the same parity.)

<u>Theorem</u>. Suppose $\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_m$ are transpositions. If $\tau_1 \cdots \tau_n = \sigma_1 \cdots \sigma_m$, then $m \equiv n \pmod 2$.

<u>Pf.</u> Notice that since σ_i's are transposition, $\sigma_i^{-1} = \sigma_i$ for every i. Hence $(\sigma_1 \cdots \sigma_m)^{-1} = \sigma_m^{-1} \cdots \sigma_1^{-1} = \sigma_m \cdots \sigma_1$. Thus $\tau_1 \cdots \tau_n = \sigma_1 \cdots \sigma_m$ implies that $\tau_1 \cdots \tau_n \sigma_m \cdots \sigma_1 = \mathrm{id}$.

Notice that $m \overset{2}{\equiv} n$ if and only if $m+n \overset{2}{\equiv} 0$. Hence

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If we show that identity cannot be written as a product of an odd number of transpositions, then

$\underbrace{\tau_1 \cdots \tau_n \sigma_m \cdots \sigma_1}_{m+n \text{ transpositions}} = id.$ implies that $m+n$ is even. So $m \overset{2}{\equiv} n$.

Therefore it is enough to prove the following claim:

<u>Claim</u>. Suppose $\gamma_1, \ldots, \gamma_k$ are transpositions and $\gamma_1 \cdots \gamma_k = id$. Then $2 \mid k$.

<u>Pf of Claim.</u> We introduce a process with the following properties:

1. The number of appearance of the largest number in the cycle form of transpositions decreases.

2. The number of transpositions either stays the same or drops by 2; in either case the parity of the number of transpositions stays the same through out this process.

Notice that because of <u>1</u> at the end no transposition will be left. Hence the final number of transpositions is 0.

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Because of 2, the parity of the number of transpositions does not change. Hence the parity of the initial number k of transposition is the same as the parity of the final number of transpositions. Since at the end there are no transpositions, we conclude that k is even.

Suppose m is the largest number that appears in the (support of) transpositions γ_i's.

We want to move all the transpositions that have m in their support toward left of this multiplication.

- If two transpositions are disjoint, then they commute
- $(a,m)(a,m) = id. \longrightarrow$ • # of transpositions drop by 2 • # of m's decreases
- $(a,m)(b,m) = (a,m)(m,b) = (a,m,b)$ $= (m,b,a) = (m,b)(b,a)$ • # of transpos. stays the same. • # of m's decrease.
- $(a,b)(a,m) = (b,a)(a,m) = (b,a,m)$ $= (m,b,a) = (m,b)(b,a)$. • # of transp. stays the same • transp. that have m are more to left.

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This process will terminate at some point. At the final state we cannot have more than one transpositions with m in their support. Because all these transpositions are on the left and if there are two such transpositions, they are either identical $(m,a),(m,a)$ and we use $(m,a)(m,a)=id.$, or they are (m,a), (m,b), then we use

$$(m,a)(m,b) = (a,m)(m,b) = (a,m,b)$$
$$= (m,b,a) = (m,b)(b,a).$$

We also notice that we cannot have only one transposition with m. Because in this case $(m,a)\ \theta_2\ \theta_3 \cdots \theta_\ell$ sends m to a. (Notice that $\theta_i(m)=m$, and so

$$m \xmapsto{\theta_\ell} m \xmapsto{\theta_{\ell-1}} m \mapsto \cdots \xmapsto{\theta_2} m \xmapsto{(m,a)} a\ .)$$

This contradicts the assumption that this product is the identity. Hence at the end of this process m disappears from the involved transposition without changing the parity of the number of transpositions. This completes the pf. $\square$

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Def. An element σ of S_n is called odd if it can be written as a product of odd number of transpositions, and it is called even if it can be written as a product of even number of transpositions. We let

$$\mathrm{sgn}: S_n \to \{1,-1\}, \quad \mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

$\mathrm{sgn}(\sigma)$ is called the sign of σ.

Notice that $(\{1,-1\}, \cdot)$ is a group.

Theorem. $\mathrm{sgn}: S_n \to \{1,-1\}$ is a group homomorphism.

Pf. Suppose $\sigma, \tau \in S_n$ and $\sigma = \sigma_1 \cdots \sigma_n$, $\tau = \tau_1 \cdots \tau_m$ where $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m$ are transpositions. Notice that $\mathrm{sgn}(\sigma) = (-1)^n$ (it is 1 if n is even, and it is -1 if n is odd.) and $\mathrm{sgn}(\tau) = (-1)^m$. We also observe that $\sigma\tau = \sigma_1 \cdots \sigma_n \tau_1 \cdots \tau_m$ can be written as a prod. of $m+n$ many transpositions. Hence $\mathrm{sgn}(\sigma\tau) = (-1)^{m+n}$. Because $(-1)^{m+n} = (-1)^m(-1)^n$, we obtain that $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$. This completes the proof. ∎

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Notice that $\ker(\mathrm{sgn}) = \{\sigma \in S_n \mid \mathrm{sgn}(\sigma) = 1\}$

$$= \{\sigma \in S_n \mid \sigma \text{ is even}\}$$

Hence $\{\sigma \in S_n \mid \sigma \text{ is even}\}$ is a subgroup of S_n.

This subgroup is called the alternating group, and it is denoted by A_n.

<u>Ex.</u> Suppose $\sigma = (a_1, \dots, a_m)$ is an m-cycle. When is σ odd or even?

<u>Solution.</u> Using the linking relation we have

$$(a_1, \dots, a_m) = (a_1, a_2)(a_2, a_3) \cdots (a_{m-1}, a_m).$$

Hence an m-cycle is a product of $m-1$ transpositions.

Therefore an m-cycle is even exactly when m is odd. ∎

<u>Ex.</u> The parity of σ and its conjugates are the same.

<u>Solution.</u> For every $\tau \in S_n$,

$$\mathrm{sgn}(\tau\sigma\tau^{-1}) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)^{-1}$$ (because sgn is a group hom.)

$$= \mathrm{sgn}(\sigma)$$ $\{1,-1\}$ is abelian and so $\mathrm{sgn}(\tau)\,\mathrm{sgn}(\tau)^{-1}\,\mathrm{sgn}(\sigma)$

Hence σ and $\tau\sigma\tau^{-1}$

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have the same parity.

Ex. Suppose σ is odd. Show that $\tau\sigma\tau$ is also odd.

Pf. $\operatorname{sgn}(\tau\sigma\tau) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

$= \operatorname{sgn}(\tau)^2 \operatorname{sgn}(\sigma)$ ($\{1,-1\}$ abelian)

$= \operatorname{sgn}(\sigma)$ ($(\pm 1)^2 = 1.$)

Ex. For every $\sigma, \tau \in S_n$, $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Pf. $\operatorname{sgn}(\sigma\tau\sigma^{-1}\tau^{-1}) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)^{-1}\operatorname{sgn}(\tau)^{-1}$

($\{1,-1\}$ is abelian) $= \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma)^{-1}\operatorname{sgn}(\tau)\operatorname{sgn}(\tau)^{-1}$

$= 1.$