Math 103B Midterm 1 Solution

Haiyu Huang

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1

Make the computation in the following ring.

(a) $(13,3)(12,24)$ in $\mathbb{Z}_{26} \times \mathbb{Z}_{48}$.

$$(13,3)(12,24) = (26 \cdot 6, 2 \cdot 24 + 24) = (0,24).$$

(b) $(1-3x)^{-1}$ in $\mathbb{Z}_{27}[x]$.

Note that $(3x)^3 = 27x^3 = 0$. So,

$$1 = 1-(3x)^3 = (1-3x)(1+3x+(3x)^2) = (1-3x)(1+3x+9x^2).$$

So, $(1-3x)^{-1} = 1+3x+9x^2$.

(c) $(3^{-1})(2)$ in $\mathbb{Z}_{11}$.

$3 \cdot 4 = 12 = 1$ implies $3^{-1} = 4$. So $(3^{-1})(2) = 4 \cdot 2 = 8$.

2

Find the characteristic of the following ring. Justify your answer.

(a) $\mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{15}$.

Since it is a finite unital ring, $\text{char}(\mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{15})$ is the additive order of $(1,1,1)$. $m(1,1,1) = 0$ iff $6 \mid m, 10 \mid m, 15 \mid m$ iff $\text{lcm}(6,10,16) = 30 \mid m$. Hence, $\text{char}(\mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{15}) = 30$.

(b) $2\mathbb{Z}_6$.

$3(2\mathbb{Z}_6) = 6\mathbb{Z}_6 = 0$ so $\text{char}(2\mathbb{Z}_6) \mid 3$. Since 3 is prime and $2\mathbb{Z}_6 \neq 0$, $\text{char}(2\mathbb{Z}_6) = 3$. Alternatively, we can find the least common multiple of the additive orders of all the elements in $2\mathbb{Z}_6 = \{0,2,4\}$.

3

Suppose D is a finite field.

(a) Prove the characteristic of D is prime.

Proof. Since D is a finite unital ring, $\text{char}(D) = \text{ord}(1) < \infty$. Suppose to the contrary that $\text{ord}(1) = ab$ for some $1 < a, b < \text{ord}(1)$. Then

$$0 = (ab)1 = (\underbrace{1+1+\cdots+1}_{a \text{ times}})(\underbrace{1+1+\cdots+1}_{b \text{ times}}) = (a1)(b1).$$

Since D is a finite field, it is an integral domain so $a1 = 0$ or $b1 = 0$, contradicting $\text{ord}(1) = ab$. ∎

(b) Suppose $\mathrm{char}(D) = p$. Prove that $f : D \to D$, $f(x) = x^p$ is a ring isomorphism. (You do not need to prove $p \mid \binom{p}{i}$ for $0 < i < p$.)

Proof.
- f is a ring homomorphism:

$$f(xy) = (xy)^p = x^p y^p = f(x)f(y)$$

by commutativity and

$$f(x+y) = (x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^i y^{p-i} = x^p + y^p = f(x) + f(y).$$

- f is injective: It suffices to show the kernel of f is trivial. $x \in \ker f \iff f(x) = x^p = 0 \iff x = 0$ as D has no zero-divisors.
- f is surjective: Since f is finite and $f : D \to D$ is injective, by the pigeonhole principle it is surjective.

Hence f is an isomorphism. ■

4

$\mathbb{Q}[\sqrt{3}]$ is a subring of $\mathbb{R}$. Show it is a field.

Proof. It suffices to show any nonzero element is invertible. Let $a + b\sqrt{3} \in \mathbb{Q}[\sqrt{3}] \setminus \{0\}$. Then $a - b\sqrt{3} \neq 0$ as $\sqrt{3} \notin \mathbb{Q}$ and $a \neq 0$ or $b \neq 0$. So $a^2 - 3b^2 = (a + b\sqrt{3})(a - b\sqrt{3}) \neq 0$. Since $\mathbb{R}$ is a field, $\frac{1}{a+b\sqrt{3}} \in \mathbb{R}$ exists and

$$\frac{1}{a+b\sqrt{3}} = \frac{1}{a+b\sqrt{3}} \cdot \frac{a-b\sqrt{3}}{a-b\sqrt{3}} = \frac{a}{a^2-3b^2} + \frac{-b}{a^2-3b^2}\sqrt{3} \in \mathbb{Q}[\sqrt{3}].$$

Therefore, $\mathbb{Q}[\sqrt{3}]$ is a field. ■