1. Suppose \( E \) is a finite field. Prove \( \prod_{\alpha \in E \setminus \{0\}} (x-\alpha) = (-1)^{|E|} \).

   (Hint. Suppose \( |E| = q \). Use \( x^q - x = \prod_{\alpha \in E} (x-\alpha) \).

2. Suppose \( p \) is prime, \( n \in \mathbb{Z}^+ \), \( p \nmid n \), and \( E \) is a field of characteristic \( p \). Prove that \( x^n - 1 \) does not have a zero with multiplicity more than 1.

3. Suppose \( f(x) \in \mathbb{Z}_p[x] \) is irreducible of degree \( n \). Prove that \( f(x) \mid x^n - x \).

   (Hint. Let \( E := \mathbb{Z}_p[x] / \langle f(x) \rangle \), and \( \alpha := x + \langle f(x) \rangle \).

   Then \( E \) is a finite field of order \( q^n \). Hence \( \alpha^{q^n} = \alpha \).

   This implies \( \alpha^{q^n} - x \in \langle f(x) \rangle \).

   (In class, I took a more advanced route.)

4. Suppose \( f(x) \in \mathbb{Z}_p[x] \) is of positive degree. Prove that \( f(x) \mid x^k - x \) for some \( k \in \mathbb{Z}^+ \) if \( f(x) \) is not divisible by the square of an irreducible poly.

   (Hint. Write \( f(x) \) as a product of irreducible; use problem 3; use \( x^{m^n} - x \mid x^{k^n} - x \) if \( m \mid n \).)
5. Prove that \( \mathbb{Z}_3[x]/\langle x^3 - x + 1 \rangle \cong \mathbb{Z}_3[x]/\langle x^3 - x + 2 \rangle \)

(Hint: Prove that both sides are fields of order \( 3^3 = 27 \).)

6. Let \( \mathbb{Q}(e^{\frac{2\pi i}{n}}) \) be the smallest subfield of \( \mathbb{C} \) that contains \( \mathbb{Q} \) and \( e^{\frac{2\pi i}{n}} \). Prove that \( \mathbb{Q}(e^{\frac{2\pi i}{n}}) \) is a splitting field of \( x^n - 1 \) over \( \mathbb{Q} \).