1. a) Find all the solutions of $x^2 - x - 2$ in $\mathbb{Z}_{17}$.
   
   b) Does $x^2 - x - 2$ have only two zeros in $\mathbb{Z}_{18}$?

2. Find the characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6$ and $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9$.
   (Justify your answer.)

3. a) Show that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field.
   
   b) Similarly one can show that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a ring. Prove that $\mathbb{Q}[\sqrt{2}]$ is the field of fractions of $\mathbb{Z}[\sqrt{2}]$ (up to an isomorphism).

4. Let $f: \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Q}[a \begin{bmatrix} 2b \\ a \end{bmatrix} | a, b \in \mathbb{Z}]$,
   
   $f(a + \sqrt{2}b) = \begin{bmatrix} a \\ b \end{bmatrix}$ is an isomorphism of rings.
   
   (You do not need to show that the codomain is a subring of $M_2(\mathbb{Z})$.)

5. Suppose $A$ is a unital commutative ring of characteristic $p > 0$, where $p$ is prime. Prove that, for any $x, y \in A$, $(x+y)^p = x^p + y^p$.
   (Hint: You are allowed to use binomial expansion without proof.)
\[(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \] where \[\binom{n}{i} = \frac{n!}{i! (n-i)!}\], and \[\binom{n}{i} \in \mathbb{Z}\].

Prove \[p \mid \binom{p}{i}\] if \[0 < i < p\] and \[p\] is prime.

Deduce the claim.

6(a) Find a zero-divisor in \[\mathbb{Z}_5[i] = \{a+bi \mid a, b \in \mathbb{Z}_5\}\]

where \[(a+bi)(c+di) = (ac-bd) + (ad+bc)i\].

(You do not need to show it is a ring.)

6(b) Show that \[x^2 + 1\] has no zero in \[\mathbb{Z}_7\].

6(c) Show that, if either \(a \neq 0\) or \(b \neq 0\) in \[\mathbb{Z}_7\], then \[a^2 + b^2 \neq 0\] in \[\mathbb{Z}_7\].

6(d) Show that \[\mathbb{Z}_7[i] = \{a+bi \mid a, b \in \mathbb{Z}_7\}\] is a field.

(Hint. 6(c) if \(a \neq 0\), then \[a^2 + b^2 = a^2 \left(1 + \left(\frac{b}{a}\right)^2\right)\]; use 6(b).

6(d) It is enough to show \[\mathbb{Z}_7[i]\] is an integral domain. (why?)

\[(a+bi)(c+di) = 0 \Rightarrow (a+bi)(a-bi)(c-di) = 0 \Rightarrow (a^2+b^2)(c^2+d^2) = 0\] in \[\mathbb{Z}_7\] use 6(c) to show either \(a+bi = 0\) or \(c+di = 0\).