1. (a) Suppose $p$ is a prime number. Prove that $x^p - x + 1$ has no zero in $\mathbb{Z}_p$.

(b) Prove that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3 [x]$.

(Remark. Using Galois theory one can show that $x^p - x + 1$ is irreducible in $\mathbb{Z}_p [x]$ for any prime $p$.)

2. (a) Prove that $x^3 - 2$ is irreducible in $\mathbb{Q} [x]$.

(b) Let $\phi_{3/2} : \mathbb{Q} [x] \rightarrow \mathbb{R}$, $\phi_{3/2} (f(x)) = f(\sqrt[3]{2})$ be the evaluation at $\sqrt[3]{2}$. We know that $\phi_{3/2}$ is a ring homomorphism.

(b-1) Prove that $\ker \phi_{3/2} = \langle x^3 - 2 \rangle$.

(b-2) Prove that $\text{Im } \phi_{3/2} = \{ a_0 + \sqrt[3]{2} a_1 + \sqrt[3]{4} a_2 | a_0, a_1, a_2 \in \mathbb{Q} \}$.

(b-3) Let $\mathbb{Q} [\sqrt[3]{2}] := \{ a_0 + \sqrt[3]{2} a_1 + \sqrt[3]{4} a_2 | a_0, a_1, a_2 \in \mathbb{Q} \}$.

Prove that $\mathbb{Q} [x] / \langle x^3 - 2 \rangle \simeq \mathbb{Q} [\sqrt[3]{2}]$.

(b-4) Prove that $\mathbb{Q} [\sqrt[3]{2}]$ is a field.

3. (a) Prove that $\sqrt{-21}$ is irreducible in $\mathbb{Z} [\sqrt{-21}]$.

(b) Prove that $\langle \sqrt{-21} \rangle$ is not a prime ideal of $\mathbb{Z} [\sqrt{-21}]$.

(c) Prove that $\mathbb{Z} [\sqrt{-21}]$ is not a PID.
4. Let \( \omega := \frac{-1+\sqrt{-3}}{2} \). Notice that \( \omega^2 + \omega + 1 = 0 \); and so
\[ \omega + \overline{\omega} = -1 \quad \text{and} \quad \omega \overline{\omega} = 1 \] where \( \overline{\omega} \) is the complex conjugate of \( \omega \). Let \( \mathbb{Z}[\omega] := \{ a + b\omega \mid a, b \in \mathbb{Z} \} \). We know that \( \mathbb{Z}[\omega] \) is a subring of \( \mathbb{C} \). Let \( \mathbb{Q}[\omega] := \{ a + b\omega \mid a, b \in \mathbb{Q} \} \).

(a) Prove that \( \mathbb{Q}[x]/\langle x^2 + x + 1 \rangle \cong \mathbb{Q}[\omega] \) and \( \mathbb{Q}[\omega] \) is a field.

(b) Prove that for any \( z \in \mathbb{Q}[\omega] \) there is \( u \in \mathbb{Z}[\omega] \) such that
\[ |z - u| \leq \sqrt{3}/3. \]

(Hint. Use the following figure.)

(Only in this part it is OK to be pictorial)

(c) Prove that for any \( a \in \mathbb{Z}[\omega] \) and \( b \in \mathbb{Z}[\omega] \setminus \{\omega, 3\} \), there are \( q, r \in \mathbb{Z}[\omega] \) such that
\[ a = bq + r \quad \text{and} \quad |r| \leq \frac{\sqrt{3}}{3} |b|. \]

(Hint. Consider \( \frac{a}{b} \in \mathbb{Q}[\omega] \); use part (b) to find \( q \in \mathbb{Z}[\omega] \).
\[ |\frac{a}{b} - q| \leq \frac{\sqrt{3}}{3}. \] Let \( r := b \left( \frac{a}{b} - q \right) \).

(d) Prove that \( \mathbb{Z}[\omega] \) is a Euclidean domain. (Hint. Let \( N(a) := |a|^2 \).)

(e) Prove that \( \mathbb{Z}[\omega] \) is a PID.
5. Suppose $a, b \in \mathbb{Z}$ and $a^2 + ab + b^2 = p$ is a prime number $> 3$.

(a) Prove that $a - b\omega$ is irreducible in $\mathbb{Z} [\omega]$.

(Hint. Consider $|a - b\omega|^2$.)

(b) Prove that $\exists \alpha \in \mathbb{Z}_p$ such that

\begin{align*}
\text{(b-1) } & \alpha^2 + \alpha + 1 = 0 \quad \text{in } \mathbb{Z}_p, \\
\text{(b-2) } & a - b\alpha = 0 \quad \text{in } \mathbb{Z}_p.
\end{align*}

(c) Let $\phi : \mathbb{Z} [\omega] \rightarrow \mathbb{Z}_p$, $\phi (c + d\omega) := c + d\alpha$

where $\alpha$ is given in part (b). Prove that $\phi$ is a ring homomorphism.

(d) Prove that $\ker \phi = \langle a - b\omega \rangle$.

(Hint. Use problems 4.e and 5.a.)

(e) Prove that $\mathbb{Z} [\omega] / \langle a - b\omega \rangle \cong \mathbb{Z}_p$. 