1. Prove that the following polynomials are irreducible.

(a) \( x^n - 12 \in \mathbb{Q}[x] \) if \( n \geq 2 \).

(b) \( x^3 - 3x^2 + 3x + 4 \in \mathbb{Q}[x] \)

(c) We are told that \( x^p - x + a \) is irreducible in \( \mathbb{Z}_p[x] \) if \( p \) is prime and \( a \in \mathbb{Z}_p \setminus \{0\} \). Use this fact only for this part of this problem.

\[ x^5 - 10x^3 + 25x^2 - 51x + 2017 \in \mathbb{Q}[x]. \]

(d) \( x^4 + 3x^3 + 27x - 12 \in \mathbb{Q}[x] \).

(e) \( x^5 - x + 1 \in \mathbb{Z}_3[x] \)

(First show it has no zero in \( \mathbb{Z}_3 \). Next you can use the following fact without proof: the only monic degree 2 polynomials in \( \mathbb{Z}_3[x] \) that do not have a zero in \( \mathbb{Z}_3 \) are \( x^2 + 1 \), \( x^2 + x - 1 \), and \( x^2 - x - 1 \).)

(f) \( x^5 + 2x + 4 \in \mathbb{Q}[x] \)

(Use part (e).)
2. Prove that $\mathbb{Z}_3[x]/\langle x^5 - x + 1 \rangle$ is a field of order $3^5$.

(Hint. (1) Use problem 1(e) to show it is a field.

(2) Use long division to show any element of this ring has a unique expression of the form:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \langle x^5 - x + 1 \rangle$$

for some $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Z}_3$.)