1. (a) \( \mathbb{Z}_3 \) is a field, \( x^3 - x + 1 \) has degree 3.

\( x^3 - x + 1 \) is irreducible in \( \mathbb{Z}_3[x] \) iff \( x^3 - x + 1 \) has no root in \( \mathbb{Z}_3 \).

Since \( 0^3 - 0 + 1 = 1 \neq 0 \)
\( 1^3 - 1 + 1 = 1 \neq 0 \)
\( 2^3 - 2 + 1 = 7 \neq 0 \)

\( \Rightarrow x^3 - x + 1 \) has no root in \( \mathbb{Z}_3 \).

(b) \( \mathbb{Z}_3[x] \) is a PID and \( x^3 - x + 1 \) is irreducible in \( \mathbb{Z}_3[x] \)

\( \Rightarrow <x^3 - x + 1> \) is maximal ideal

\( \Rightarrow \mathbb{Z}_3 / <x^3 - x + 1> \) is a field

(c) We know that \( \mathbb{Z}_3 [x] / <x^3 - x + 1> \) has \( 3^3 = 27 \) elements.

Consider the map \( \phi_a : \mathbb{Z}_3 [x] \rightarrow \mathbb{C} \)
\( g(x) \rightarrow g(a) \)

\( \text{Im} \phi_a = \{ c_0 + c_1 a + c_2 a^2 \mid c_i \in \mathbb{Z}_3 \} \).

Clearly \( c_0 + c_1 a + c_2 a^2 \in \text{Im} \phi_a \).

Now for any \( g(x) \in \mathbb{Z}_3 [x] \), \( g(x) = p(x) (x^3 - x + 1) + r(x) \), \( p(x), r(x) \in \mathbb{Z}_3 [x] \), \( \deg r(x) \leq 2 \).

Then \( g(a) = p(a) \cdot 0 + r(a) \in \{ c_0 + c_1 a + c_2 a^2 \mid c_i \in \mathbb{Z}_3 \} \).

2 is a root of \( x^3 - x + 1 \) \( \Rightarrow <x^3 - x + 1> \subseteq \text{Ker} \phi_a \).

But \( 1 \notin \text{Ker} \phi_a \Rightarrow \text{Ker} \phi_a \neq \mathbb{Z}_3 [x] \Rightarrow <x^3 - x + 1> \neq \text{Ker} \phi_a \).

By 1st Isomorphism Theorem, \( \mathbb{Z}_3 [x] / <x^3 - x + 1> \cong \{ c_0 + c_1 a + c_2 a^2 \mid c_i \in \mathbb{Z}_3 \} \).
\( \Rightarrow \{ c_0 + c_1 a + c_2 a^2 \mid c_i \in \mathbb{Z}_3 \} \) is field of 27 elements, with root of \( x^3 - x + 1 \), which is 2.

2. (a) Consider 3, prime number

\( 3 \mid 6, 3 \mid 30, 3 \mid 12 \), but \( 3^2 + 12 \)

\( \Rightarrow f(x) \) is irreducible by Eisenstein criteria.

(b) Consider the evaluation map \( \phi_a : \mathbb{R}[x] \rightarrow \mathbb{C} \)
\( g(x) \rightarrow g(a) \)

\( f(x) \) has 2 as root and \( f(x) \) is irreducible

\( \Rightarrow \text{Ker} \phi_a = <f(x)> \).
By the main theorem of evaluation map, we have since deg \( f(x) = 5 \)
\[ \text{Im} \phi = \{ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \mid c_i \in \mathbb{Q} \} \] and the image is a field.

(2) Suppose we have \( a_0 + a_1 x + \cdots + a_4 x^4 = 0, \ a_i \in \mathbb{Q} \).

Consider \( g(x) = a_0 + a_1 x + \cdots + a_4 x^4, \ g(2) = 0 \) by assumption.

\[ g(x) \in \ker \phi = \langle f(x) \rangle \implies g(x) = f(x) \cdot h(x) \]

But \( \deg g(x) \leq 4 < \deg f(x) = 5 \)

\[ \implies \text{the only possibility is that } g(x) = h(x) = 0 \]

\[ \implies a_i = 0, \ i = 0, \ldots, 4. \]

3. \( f(x) \) is irreducible in \( \mathbb{Q}[x] \) iff \( f(-x) \) is irreducible iff \( f(-1(x+1)) \) is irreducible.

\[ f(-1(x+1)) = (x+1)^5 + (x+1)^4 + \cdots + (x+1) + 1 \]

\[ = \frac{1 - (1 + x)^6}{1 - (1 + x)} = \frac{x^5 + (5) x^4 + \cdots + (5)(x+1) x + (p)}{p} \]

\[ p \mid (\psi) \] but \( p^2 \nmid (\psi) \)

By Eisenstein's criteria, it's irreducible.

Another way of writing 3:

\( f(x) \) is irreducible iff \( f(-x) \) is irreducible.

Let \( g(x) = f(-x) = x^5 + x^4 + \cdots + x + 1 \).

As we did in Lecture, \( g(x) = \frac{x^5 + 1}{x + 1} \)

\[ g(y+1) = \frac{(y+1)^5 + 1}{(y+1) + 1} = \frac{y^5 + 5y^4 + \cdots + 5y + 1}{y} = y^4 + (5) y^3 + \cdots + (5) \]

\( g(y+1) \) is irreducible by Eisenstein's criteria. \( (p) \mid (\psi), 1 < \psi, p^2 \nmid (\psi) \).

Suppose \( g(x) \) is reducible, then

\[ g(x) = g_1(x) g_2(x), \ \text{with deg } g_i(x) \geq 1. \]

\[ \implies g(y+1) = g_1(y+1) g_2(y+1), \ \text{with deg } g_i(y+1) \geq 1, \ \text{contradiction.} \]

\[ \implies g(x) = f(-x) \text{ is irreducible} \]

\[ \implies f(x) \text{ is irreducible.} \]
4. (a) \( x^4 - 2x^2 - 2 = (\sqrt{1+\sqrt{5}})^4 - 2(\sqrt{1+\sqrt{5}})^2 - 2 \)
\[= (1+\sqrt{5})^2 - 2(1+\sqrt{5}) - 2 = 1 + 2\sqrt{5} + 5 - 2 - 2\sqrt{5} - 2 = 0. \]
\[\Rightarrow 2 \text{ is a root of } x^4 - 2x^2 - 2 \]
By Eisenstein's criteria, we notice that \( 2 \mid b \text{ but } 2^2 \nmid a \).
\[\Rightarrow x^4 - 2x^2 - 2 \text{ is irreducible} \]
\[\Rightarrow x^4 - 2x^2 - 2 \text{ is minimal polynomial} \]
(b) As usual, consider the evaluation map \( \phi_a \) at \( 2 \).
\( x^4 - 2x^2 - 2 \) is irreducible and admits \( 2 \) as a root.
\[\Rightarrow \ker \phi_a = \langle x^4 - 2x^2 - 2 \rangle \]
By the main theorem of evaluation map and the fact that \( \deg x^4 - 2x^2 - 2 = 4. \)
We have that \( \text{Im} \phi = \{ a_0 + a_1 2 + a_2 2^2 + a_3 2^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Q} \} \) is a field.

5. \( x^2 + 2 \) is irreducible in \( \mathbb{Z}_5[x] \) as it has no root.
\[\Rightarrow \mathbb{Z}_5[x]/\langle x^2 + 2 \rangle \text{ is a field with } 5^2 = 25 \text{ elements}. \]