Homework 6, math103b winter 2019

1. Suppose E is a field extension of \( \mathbb{Z}_3 \) that contains a zero \( \alpha \) of \( x^3 - x + 1 \).

   (a) Prove that \( \mathbb{Z}_3[\alpha] = \{ a_0 + a_1 \alpha + a_2 \alpha^2 | a_0, a_1, a_3 \in \mathbb{Z}_3 \} \).

   (b) Prove that \( \mathbb{Z}_3[\alpha] \) is a field and \( |\mathbb{Z}_3[\alpha]| = 27 \).

2. Let \( I = \{ 2p(x) + xq(x) | p(x), q(x) \in \mathbb{Z}[x] \} \). Prove that \( I \) is not a principal ideal and deduce that \( \mathbb{Z}[x] \) is not a PID.

3. Let \( \beta := \sqrt{1 + \sqrt{3}} \).

   (a) Prove that the minimal polynomial \( m_\beta(x) \) of \( \beta \) over \( \mathbb{Q} \) is \( x^4 - 2x^2 - 2 \).

   (b) Prove that 

   \[
   \mathbb{Q}[\beta] = \{ a_0 + a_1 \beta + a_2 \beta^2 + a_3 \beta^3 | a_0, a_1, a_2, a_3 \in \mathbb{Q} \}.
   \]

   (c) Prove that \( \mathbb{Q}[\beta] \cong \mathbb{Q}[x]/(x^4 - 2x^2 - 2)\mathbb{Q}[x] \) and it is a field.

   (d) Write \( \beta^{-1} \) as a \( \mathbb{Q} \)-linear combination of \( 1, \beta, \beta^2, \beta^3 \).