Name: ______________________________

PID: ______________________________

Section: ______________________________

Question	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	25	
Total:	90	

1. Write your Name, PID, and Section on the front page of your exam.
2. No smart phone or other electronic devices are allowed during this exam.
3. If you have to leave the lecture hall during the exam for any reason, you have to leave your exam and smart phone or other electronic devices on the table in front of the lecture hall with the proctor.
4. Read each question carefully, and answer each question completely.
5. Write your solutions clearly in the exam sheet.
6. Show all of your work; no credit will be given for unsupported answers.
7. You may cite major results from lecture as long as the purpose of the problem is not to reproduce a special case of the same result.
8. When you are using a major result from lecture, you should give its precise statement and say why its hypothesis holds in your setting.

1. Determine if the following rings are fields or not. Justify your answer.

   (a) (5 points) $\mathbb{Z}_{10}$.

   (b) (10 points) $\mathbb{Q}[x]/\langle x^2 + 1\rangle$.

2. We know that $I := 2\mathbb{Z}[x]$ is an ideal of $\mathbb{Z}[x]$ and $\mathbb{Z}[x]/I \simeq \mathbb{Z}_2[x]$. Use this fact to answer the following questions. Justify your answers.

   (a) (5 points) Is I a prime ideal?

   (b) (5 points) Is I a maximal ideal?

3. Determine if the following polynomials have a zero in $\mathbb{Q}$. Justify your answer.

(a) (5 points) $x^7 + 2018x^5 + 103x^2 + 8x + 1$.

(b) (5 points) $x^{3^{103}} - x^3 + 2018$.

4. (10 points) State the main theorem on evaluation maps; to be precise. Let $\alpha \in \mathbb{C}$ and $\phi_\alpha : \mathbb{Q}[x] \to \mathbb{C}, \phi_\alpha(f(x)) = f(\alpha)$. We know that ϕ_α is a ring homomorphism. State carefully what the main theorem on evaluation maps tells us about the kernel and image of ϕ_α.

5. (10 points) Suppose $\alpha \in \mathbb{C}$ is a zero of $x^5 - 2x^2 - 8x - 18$. Let

$$\phi_\alpha : \mathbb{Q}[x] \to \mathbb{C}, \phi_\alpha(f(x)) = f(\alpha).$$

We know that ϕ_α is a ring homomorphism. Prove that

$$\ker \phi_\alpha = \langle x^5 - 2x^2 - 8x - 18 \rangle.$$

6. (10 points) Suppose $\mathbb{F}_q$ is a finite field of order q. Prove that

$$x^q - x = \prod_{\alpha \in \mathbb{F}_q} (x - \alpha).$$

(Hint: For any finite group G and $g \in G$, we have $g^{|G|} = 1$.)

7. Let $\mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} | \, a, b \in \mathbb{Z}\}$. We know that $\mathbb{Z}[\sqrt{-6}]$ is an integral domain. Let us also recall that $|a + b\sqrt{-6}|^2 = a^2 + 6b^2$ and using properties of norm of complex numbers we know that $|(a + b\sqrt{-6})(c + d\sqrt{-6})|^2 = |a + b\sqrt{-6}|^2|c + d\sqrt{-6}|^2$. Furthermore you are allowed to use without a proof that $U(\mathbb{Z}[\sqrt{-6}]) = \{-1, 1\}$.

   (a) (3 points) Show that, for $a, b \in \mathbb{Z}$, $a^2 + 6b^2 < 6$ implies $b = 0$. Deduce that $|a + b\sqrt{-6}|^2 \notin \{2, 3\}$ for $a, b \in \mathbb{Z}$.

   (b) (2 points) Show that $|a + b\sqrt{-6}|^2 = 1$ if and only if $a + b\sqrt{-6} \in \{1, -1\}$, for $a, b \in \mathbb{Z}$.

(c) (5 points) Show that $\sqrt{-6}$ is irreducible in $\mathbb{Z}[\sqrt{-6}]$.

(d) (5 points) Show that $\langle\sqrt{-6}\rangle$ is not a prime ideal of $\mathbb{Z}[\sqrt{-6}]$.

(e) (10 points) Show that $\mathbb{Z}[\sqrt{-6}]$ is not a PID.

Good Luck!