Math 103B - HW-2 (solution)

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All problems are from **A first course in Abstract Algebra** by John B. Fraleigh.

Chapter 19

24 Show that an intersection of subdomains of an integral domain D is again a subdomain of D.

Proof. Let A_i's be subdomains of D, where i lies in some indexing set I and $A = \cap A_i$. Note that $1 \in A_i$ for all $i \in I$, $1 \in A$. Moreover A is a subring since for any $a, b \in A$, $a + b$ and ab are in all A_i hence in A. Commutativity follows from the commutativity of D, so we are left to show that it does not have zero-divisors. Suppose $a \in A$ is a zero divisor, then there is $b \in A$ such that $a.b = 0$. Since $a, b \in D$ this is not possible. Thus A is a subdomain of D. $\square$

27 Show that the characteristic of a subdomain of an integral domain D is equal to the characteristic of D.

Proof. Let A be a subdomain of D, then unity $1 \in A$.

If characteristic of D is 0, then $n \cdot 1 := 1 + 1 + \cdots + 1$ sum of n ones is not zero, otherwise $n \cdot a = (n \cdot 1) \cdot a = 0$. Thus $m \cdot 1 \neq 0$ in A for all $m \in \mathbb{Z}$, which implies char of A is 0.

If characteristic of D is prime p, then for any $a \in A$, $p \cdot a = (p \cdot 1) \cdot a = 0$, thus p is the characteristic of A. $\square$

28 Show that if D is an integral domain, then $A = \{n \cdot 1 | n \in \mathbb{Z}\}$ is a subdomain of D contained in every sub domain of D.

Proof. Let p be the characteristic of D. Note that A is closed under addition and multiplication where multiplication induced by D can be observed to be $(n \cdot 1)(m \cdot 1) = (mn \mod p) \cdot 1$. Thus A is a subring, moreover is contains unity and it is commutative. $\square$

Chapter 21

2 Describe the field F of quotients of the integral subdomain $D = \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$ of $\mathbb{R}$.

Proof. Let $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} : r, s \in \mathbb{Q}\}$, we have seen that it is a field contained in $\mathbb{R}$, since

$$(r + s\sqrt{2})^{-1} = \frac{r - s\sqrt{2}}{r^2 - 2s^2}$$

where the denominator is non-zero since 2 is an irrational number.

We claim that field of quotient of D in $\mathbb{R}$ is precisely $\mathbb{Q}[\sqrt{2}]$. This follows since cearly $D \subset \mathbb{Q}[\sqrt{2}]$, and for any element $r + s\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ there exist an integer N such that Nr and Ns are integers, thus

$$r + s\sqrt{2} = \frac{Nr + Ns\sqrt{2}}{N}$$

where both the denominator and the numerator is in D. $\square$

4 Mark each of the following true or false:
(a). True : By definition.
(b). False : existence of irrational numbers.
(c). True : Because $\mathbb{R}$ is a field
(d). false : since $\mathbb{R}$ is a field strictly contained in $\mathbb{C}$
(e). True : By universal property there is a $\psi : Q(D) \to D$ which is injection, since D is a field. Since ψ takes D to D, it sujection, hence isomorphism.
(h). True : By definition.
(i). True : Let F be field of quotients of D, then $D' \subset D \subset F$, so by universal property of F', there is an injection $\phi : F' \to F$. So F' is isomorphic to the image of F' via ϕ which is a subfield of F.
(j). True : Any two field of quotients are isomorphic.

5 Show by example that a field F' of quotients of a proper subdomain D' of an integral domain D may also be a field F of quotients for D.

Proof. We have plenty of possible solutions, I will state a few :
(i) $D = \mathbb{Q}$, $D' = \mathbb{Z}$, so $F = \mathbb{Q} = F'$
(ii) $D = \mathbb{Z}[\frac{1}{n}]$, $D' = \mathbb{Z}$, so $F = \mathbb{Q} = F'$ for any positive integer n.
(iii) $D = \mathbb{Z}[i]$ $D' = \mathbb{Z}[2i]$, then $F = \mathbb{Q}[i] = F'$ $\square$

Chapter 22

5 How many polynomial are there of degree ≤ 3 in $\mathbb{Z}_2[x]$?

Answer. Any polynomial of degree at most 3 can be written as $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where $a_i \in \mathbb{Z}_2$. Since there are 4 variables, where each one can take value 0 or 1, the total number of possible polynomial equals $2^4 = 16$. $\square$

8 Let $F = E = \mathbb{C}$, Compute the indicated evaluation homomorphism $\phi_i(2x^3 - x^2 + 3x + 2)$.

Answer. We just have to evaluate the polynomial at $x = i$, so we get

$$\begin{aligned} \phi_i(2x^3 - x^2 + 3x + 2) &= (2i^3 - i^2 + 3i + 2) \\ &= 3 + i \end{aligned}$$

$\square$

11 Let $F = E = \mathbb{Z}_7$, Compute the indicated evaluation homomorphism $\phi_4(3x^{106} + 5x^{99} + 2x^{53})$.

Answer. By Fermat's theorem, $4^6 = 1$ in $\mathbb{Z}_7$, moreover one may realise that $4^3 = 64 = 1$ in $\mathbb{Z}_7$, thus we get that $4^{106} = 4^1 = 4$, $4^{99} = 1$ and $4^{53} = 2$ in $\mathbb{Z}_7$. Thus

$$\phi_4(3x^{106} + 5x^{99} + 2x^{53}) = 3 \times 4 + 5 \times 1 + 2 \times 2 = 0 \in \mathbb{Z}_7$$

$\square$

25 Let x an indeterminate.
(b) Find the units in $\mathbb{Z}[x]$
(c) Find units in $\mathbb{Z}_7[x]$.

Answer. Note that both $D = \mathbb{Z}$ and $D = \mathbb{Z}_7$ are integral domain, thus if $f(x)$ and $g(x)$ are polynomials over D of degree n and m respectively, then degree of $f(x) \cdot g(x)$ is $n + m$.

Let $f(x)$ be a unit, then there is $g(x)$ such that $f(x) \cdot g(x) = 1$ as polynomial. By the additivity of the degree we conclude that degree of $f(x)$ and $g(x)$ equals 0. Hence $f(x) = a_0$ and $g(x) = b_0$ where $a_0, b_0 \in D$. Since $a_0 \cdot b_0 = 1$, we get that $f(x) = a_0$ is a unit in D (considered as a subdomain of $D[x]$).

Thus in part (b) we have $(\mathbb{Z}[x])^\times = \mathbb{Z}^\times = \{1, -1\}$ and in part (c) we have $(\mathbb{Z}_7[x])^\times = \mathbb{Z}_7^\times = \{1, 2, 3, 4, 5, 6\}$ where these constants are considered as polynomials. $\square$