Instead of directly proving the third part of the theorem, we will investigate the following questions:

Suppose $R$ is a unital commutative ring. Under what conditions on an ideal $I$ of $R$ do we have that $R/I$ is a field? Under what conditions on $I$ do we have that $R/I$ is an integral domain?

Investigation.

Since $R$ is a unital commutative ring, so is $R/I$. So $R/I$ is a field $\iff 1$. $R/I$ is not the zero ring; that means it has at least two elements.

1. $U(R/I) = (R/I) \setminus \{0 + I\}$.

$\iff 1. R \neq I$.

2. $x + I \neq 0 + I$ implies $\exists r \in R$ such that $r(x + I) = 1 + I$.

$\iff 1. R \neq I$

2. $x \notin I$ implies $\exists y \in I, re \in R$, $rx + y = 1$. 
\[ \Rightarrow (1) \ I \neq R \]

\[ (2) \ \forall x \in R \setminus I, \text{ if } J \trianglelefteq R, \ x \in J, \ I \subseteq J, \text{ then } 1 \in J \]

\[ \Rightarrow (1) \ I \neq R \]

\[ (2) (J \trianglelefteq R \text{ and } I \nsubseteq J) \Rightarrow J = R. \]

**Def.** We say I is a maximal ideal of R if

\[ (1) \ I \text{ is a proper ideal; this means } I \neq R \]

\[ (2) \text{ if } J \trianglelefteq R \text{ and } I \nsubseteq J, \text{ then } J = R. \]

**Theorem.** Let R be a unital commutative ring. Then

I is a maximal ideal of R if and only if \( R/I \) is a field.

**Pf.** \( (\Rightarrow) \) we have already proved.

\( (\Leftarrow) \) Since I is a proper ideal, \( R/I \) is a non-zero ring.

\[ \forall x + I \in (R/I) \setminus \{0 + I\} \Rightarrow x \in R \setminus I \]

\[ \Rightarrow \text{ the ideal generated by } \langle x^g \rangle \cup I \]

is R as I is a maximal ideal.

**Claim.** \( \langle x^g \rangle \cup I = \{rx + y \mid r \in R, y \in I \} \).

**Pf of claim.** \( x \in \langle x^g \rangle \cup I \Rightarrow \forall r \in R, y \in I, \ rx + y \in \langle x^g \rangle \cup I \).
Maximal ideals

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And so \( \mathcal{J} = \langle x, y \rangle \in \mathcal{J} \). Next we will show that \( \mathcal{J} \) is an ideal of \( R \):

\[
(r_1 x + r_2 x + y_1 + y_2) = (r_1 + r_2) x + (y_1 + y_2) \in \mathcal{J}
\]

\( \forall r, r' \in R, y \in I \), \( r'(r x + y) = (r) x + (r'y) \in \mathcal{J} \)

Since \( R \) is unital and \( 0 \in I \), \( x \in \mathcal{J} \). And since \( (0)(x) = 0 \), \( I \subseteq \mathcal{J} \). So \( \mathcal{J} \subseteq \mathcal{J} \), which implies \( \langle x, y \rangle \subseteq \mathcal{J} \).

And the claim follows.

Since \( \langle x, y \rangle = R \), we deduce \( \exists \) \( r \in R, y \in I \) s.t. \( r x + y = 1 \). Hence \( r x + I = 1 + I \); and so

\[
(r + I)(x + I) = 1 + I.
\]

As \( R/I \) is commutative, we get that \( x + I \in U(R/I) \).

Therefore \( R/I \) is a field. \( \blacksquare \)