Let's go back to Euler's method and how we got the initial information about \( \sum_{p \leq x} \frac{1}{p} \). We considered (here we can work with \( x \in \mathbb{R}^{+} \))

\[
\sum_{n} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}
\]

\[
\Rightarrow \quad \ln \left( \sum_{n} \frac{1}{n^s} \right) = \sum_{p} - \ln \left(1 - \frac{1}{p^s} \right)
\]

\[
= \sum_{m \geq 1} \frac{1}{m \cdot p^m}
\]

\[
= \sum_{p} \frac{1}{p^s} + O(1)
\]

\[
\Rightarrow \quad \text{Since the LHS} \to \infty \text{ as } s \to 1^+,
\]

\[
\text{the RHS} \to \infty \text{ as } s \to 1^+
\]

\[
\Rightarrow \quad \sum_{p} \frac{1}{p^s} \to \infty \text{ as } s \to \infty.
\]

Dirichlet's idea was to reverse engineer this approach for

\[
\sum_{p \in \mathbb{P}_{N\alpha}} \frac{1}{p^s}
\]

where \( \mathbb{P}_{N\alpha} := \{ \text{prime numbers of the form } Nk + \alpha \text{, } k \in \mathbb{Z} \} \).

The first step is O.K.:

\[
\sum_{p \in \mathbb{P}_{N\alpha}} \frac{1}{p^s} + O(1) = \sum_{m \geq 1} \frac{1}{m \cdot p^m},
\]

This, however, is not \( \ln \) of a Euler-product-type of formula. Let's recall that in one of your homework assignments you have proved:

If \( f: \mathbb{Z}^+ \to \mathbb{C} \) is a strictly multiplicative and \( |f(n)| \leq 1 \),

\[
\ln \prod_{p \leq c} p = -1 \cdot \sum_{p \leq c} \frac{1}{p}.
\]
then \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - \frac{f(p)}{p^s}} \) for \( \text{Re}(s) > 1 \)

(and both sides are convergent.)

So in order to have a Euler-product-type of formula, one needs to have (strictly) multiplicative function. Let’s examine

\[ \sum_{p \in \mathbb{P}_{N, a}} \frac{1}{p^s} = \sum_p \frac{1_{N, a}(p)}{p^s} \]

where \( 1_{N, a}(m) = \begin{cases} 1 & m \in \mathbb{P}_{N, a} \\ 0 & m \notin \mathbb{P}_{N, a} \end{cases} \)

And it seems we need to change \( \sum \) to

\[ \sum_{p \in \mathbb{P}_{N, a}} \frac{1}{p^s} + O(1) = \sum_{p, m \geq 1} \frac{1_{N, a}(p^m)}{p^{ms}} \]

but we are still stucked as \( 1_{N, a} \) is NOT necessarily a (strictly) multiplicative function.

Notice that \( 1_{N, a} \) factors through \( \mathbb{Z}/_{NZ} \), i.e.

\( n_1 = n_2 \pmod{N} \Rightarrow 1_{N, a}(n_1) = 1_{N, a}(n_2) \).

So it is essentially a function \( \mathbb{Z}/_{NZ} \rightarrow \mathbb{C} \).

Since \( \gcd(a, N) = 1 \) and we are looking for multiplicative functions, it is better to think about \( 1_{N, a} \) as a function

\[ \left( \mathbb{Z}/_{NZ} \right)^{\times} \rightarrow \mathbb{C} \]

Dirichlet is a German mathematician and, when he spent time in Paris, he was a friend of Fourier. So he learned Fourier analysis from him. In the context that we need it implies the following:

**Theorem** For any finite abelian group \( G \),

\[ \hat{G} := \text{Hom}(G, \mathbb{S}^1) \]
forms an orthonormal basis of the vector space $\mathbb{C}[G]$ where

$$\langle \hat{f}_1, \hat{f}_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \hat{f}_1(g) \overline{\hat{f}_2(g)}.$$ 

(We will prove this later.)

**Corollary.** Suppose $G$ is a finite abelian group. Then

$$\forall g_0 \in G, \quad \mathbb{1}_{g_0} = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(g_0) \chi,$$

where $\mathbb{1}_{g_0}(g) = \begin{cases} 1 & \text{if } g = g_0, \\ 0 & \text{otherwise}. \end{cases}$

**Proof.** Since $\hat{G}$ is an orthonormal basis,

$$\mathbb{1}_{g_0} = \sum_{\chi \in \hat{G}} \langle \mathbb{1}_{g_0}, \chi \rangle \chi.$$ And

$$\langle \mathbb{1}_{g_0}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \mathbb{1}(g) \overline{\chi(g)} = \frac{\chi(g_0)}{|G|}.$$

So by the above Corollary

$$\forall m \in \left(\mathbb{Z}/N\mathbb{Z}\right)^*, \quad \mathbb{1}_{N^a}(m) = \frac{1}{\Phi(N)} \sum_{\chi \in \hat{\left(\mathbb{Z}/N\mathbb{Z}\right)^*}} \chi(a) \chi(m).$$

Any $\chi \in \left(\mathbb{Z}/N\mathbb{Z}\right)^*$ gives us a function from $\mathbb{Z}^+$ to $S^4$ (that we again denote by $\chi$):

$$\chi(m) = \begin{cases} \chi(m + N\mathbb{Z}) & \text{if } \gcd(m, N) = 1 \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to see that these are strictly multiplicative.

These are called *Dirichlet characters.*

Altogether we get

$$\sum_{p \in \mathcal{P}_{\mathbb{N}^a}} \frac{1}{p^s} + O(1) = \sum_{p, m \geq 1} \frac{1_{\mathbb{N}^a}(p^m)}{m p^{ms}}$$

$$- \sum_{\chi \in \hat{\left(\mathbb{Z}/N\mathbb{Z}\right)^*}} \frac{1}{\Phi(N)} \sum_{a} \overline{\chi(a)} \chi(p^m)$$
\[
\sum_{\substack{\chi \in (\mathbb{Z}/N\mathbb{Z})^* \setminus \chi_0 \setminus (\mathbb{Z}/m\mathbb{Z})^*}} \frac{1}{\varphi(N)} \sum_{\chi \in \mathbb{Z}/N\mathbb{Z}^*} \frac{\chi(p)^m}{m \varphi(m)} \cdot l(\chi, s) = \frac{1}{\varphi(N)} l(\chi_0, s) + \sum_{\chi \neq \chi_0} l(\chi, s)
\]

where \( \chi_0 \) is the trivial character, i.e.

\[
\chi_0(m) = \begin{cases} 
1 & \text{if } \gcd(m, N) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that \( \frac{1}{\varphi(N)} \sum_{\chi \neq \chi_0} l(\chi, s) = \sum_{\chi \neq \chi_0} \frac{1}{\varphi(N)} \sum_{p \mid m} \frac{1}{p^s} + O(1)
\]

Only finitely many primes divide \( N \)

So \( l(\chi_0, s) \xrightarrow{s \to 1^+} \infty \). Hence to get the desired result it is enough to show

\( l(\chi, s) \) stays bounded as \( s \to 1^+ \) if \( \chi \neq \chi_0 \).

For \( \Re(s) > 1 \), \( \sum_{p \mid m} \frac{\chi(p)^m}{m \varphi(m)} \) is absolutely convergent

\[
\Rightarrow l(\chi, s) = \sum_{p} \left( \sum_{m \geq 1} \frac{1}{m} \left( \frac{\chi(p)}{p^s} \right)^m \right)
\]

Since \( \left| \frac{\chi(p)}{p^s} \right| = \frac{1}{\Re(s)} < 1 \), we have

\[
- \ln \left( 1 - \frac{\chi(p)}{p^s} \right) = \sum_{m \geq 1} \frac{1}{m} \left( \frac{\chi(p)}{p^s} \right)^m \quad \text{for } \Re(s) > 1.
\]

\[
\Rightarrow l(\chi, s) = \sum_{p} - \ln \left( 1 - \frac{\chi(p)}{p^s} \right).
\]

\[
l(\chi, s) = \sum_{p} - \ln \left( 1 - \frac{\chi(p)}{p^s} \right).
\]
\[ e^{l(\chi, s)} = e^{\sum_{p} -\ln \left(1 - \frac{\chi(p)}{p^s}\right)} = \prod_{p} e^{-\ln \left(1 - \frac{\chi(p)}{p^s}\right)} = \prod_{p} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} =: L_{1}(\chi, s) \]

These are called Dirichlet $L_{1}$-functions. So to show $l(\chi, s)$ stays bounded as $s \to 1^{+}$, one should at least show:

\( \lim_{s \to 1^{+}} L_{1}(\chi, s) \) is not 0 and can be extended holomorphically to $\Re(s) > 1 - \varepsilon$.

(In fact, it will be extended to $\Re(s) > 0$.)

Before we prove \( \textcircled{\text{\(\star\)}} \), let's see why \( \textcircled{\text{\(\star\)}} \) is enough in order to get that $l(\chi, s)$ stays bounded as $s \to 1^{+}$.

Since $L_{1}(\chi, s)$ is holomorphic on $\Re(s) > 1 - \varepsilon$ and half-plane is open and simply-connected and $L_{1}(\chi, s) \neq 0$ for $s$ in $U$,

there is a holomorphic function $g: U \to \mathbb{C}$ such that $L_{1}(\chi, s) = e^{g(s)}$.

Hence $g(s) = l(\chi, s) + 2\pi i n(s)$ for $\Re(s) > 1$ and $s \in U$.

\[ n(s) \text{ is holomorphic on } \Re(s) > 1 \quad \Rightarrow \quad n(s) = n_{0} \text{ is constant.} \]

\[ \Rightarrow \quad l(\chi, s) = g(s) - 2\pi i n_{0} \text{ for } \Re(s) > 1 \text{ and } s \in U \]

\[ l \quad \text{h h implies } l(\chi, s) \]
stays bounded as $s \to 1^+$.

**Proposition.** If $X \neq X_0$, then $\sum_{n=1}^{\infty} \frac{X(n)}{n^s}$ is convergent for $\text{Re}(s) > 0$.

And it is uniformly on compact subsets of $\text{Re}(s) > 0$. Hence

$L_i(X,s)$ is holomorphic on $\text{Re}(s) > 0$.

**Proof.** Let $C(x) = \sum_{n \leq x} X(n)$. Then

$$\sum_{n \leq m} \frac{X(m)}{n^s} = \frac{C(m)}{m^s} + \int_1^{\infty} \frac{C(t)}{t^{s+1}} \, dt$$

\[
\begin{cases}
\sum_{Nk \leq n < Nk+N} X(n) = o \quad \text{as} \quad (X, X_0) = 0.
\end{cases}
\]

\[\Rightarrow \quad C(x) \leq N.
\]

\[\Rightarrow \quad \int_1^{\infty} \frac{C(t)}{t^{s+1}} \, dt \text{ is absolutely convergent for } \text{Re}(s) > 0
\]

and $\frac{C(m)}{m^s} \to 0$ for $\text{Re}(s) > 0$.

\[
\Rightarrow \sum_{n=1}^{\infty} \frac{X(n)}{n^s} \text{ is convergent.}
\]

(Ex. show why it is uniform on bounded closed subsets.)

So it is enough to show

$L_i(X, 1) \neq 0$ if $X \neq X_0$.

Let's consider

$$\prod_{X \in \left(\mathbb{Z}/\mathbb{N}\mathbb{Z}\right)^*} L_i(X, s) =: \zeta_N(s).$$

For $\text{Re}(s) > 1$, we have

$$\zeta_N(s) = \prod_{X \in \left(\mathbb{Z}/\mathbb{N}\mathbb{Z}\right)^*} \prod_p \frac{1}{1 - \frac{X(p)}{p^s}}$$

$$= \prod \left( \prod \left( 1 - \frac{X(p)}{p^s} \right)^{-1} \right)$$
\[
= \prod_p \left( \prod_{\chi} \left( 1 - \frac{\chi(p)}{p^s} \right) \right)
\]

- If \( p \mid N \), then \( \chi(p) = 0 \Rightarrow \prod_{\chi} \left( 1 - \frac{\chi(p)}{p^s} \right) = 1 \).

- If \( p \nmid N \), then \( p + NZ \in (\mathbb{Z}/N\mathbb{Z})^* \).

For any finite abelian group \( G \) and any \( g \in G \) we have
\[
\prod_{\chi \in \hat{G}} (T - \chi(g)) = (T^{\text{ord}_g} - 1)
\]
where \( \text{ord}_g \) is the order of \( g \) in \( G \).

(We will prove this later.)

Hence we have
\[
\zeta_N(s) = \prod_{p \nmid N} \left( 1 - \frac{1}{\text{ord}_N(p^s)} \right)^{-\frac{\varphi(N)}{\varphi(N) s}}
\]
for \( \Re(s) > 1 \), and if \( L(\chi, s) = 0 \) for some \( \chi \), then \( \zeta_N(s) \) is holomorphic
on \( \Re(s) > 0 \).

Let's focus on values of right hand side of the above equation at

real values for \( s \). For any \( \alpha \in \mathbb{R}^+ \),
\[
\left( 1 - \frac{1}{\text{ord}_N(p^s)} \right)^{-\frac{\varphi(N)}{\varphi(N) s}} \geq \left( 1 - \frac{1}{\varphi(N) s} \right)^{-1}
\]

\[
\Rightarrow \prod_{p \nmid N} \left( 1 - \frac{1}{\text{ord}_N(p^s)} \right)^{-\frac{\varphi(N)}{\varphi(N) s}} \geq \prod_{p \nmid N} \left( 1 - \frac{1}{\varphi(N) s} \right)
\]
\[
= \zeta(\varphi(N) s) \prod_{p \nmid N} \left( 1 - \frac{1}{\varphi(N) s} \right)
\]

\[
\Rightarrow \text{The right hand side of } \Box \text{ diverges at } s = \frac{1}{\varphi(N)}.
\]

It seems contradictory: the RHS has a holomorphic to \( \Re(s) > 0 \),

but itself diverges at \( s = \frac{1}{\varphi(N)} \). Can we get a contradiction?

So we focus on the RHS. Notice that \( (1 + x + x^2 + ...) \equiv \sum_{k=0}^{\infty} c_k x^k \)
where \( c_k \in \mathbb{R}^+ \). (We do not need this here, but it is worth mentioning)
where $c_k \in \mathbb{Z}^+$. (We do not need this here, but it is worth mentioning that there is a binomial expansion even for negative powers:

\[
(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} (-1)^k x^k \quad \text{where}
\]

\[
\binom{-m}{k} = \frac{(-m)(-m-1) \cdots (-m-k+1)}{k!} = (-1)^k \binom{m-k+1}{k}
\]

So \((1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m-k+1}{k} x^k \) for \( |x| < 1 \).

Hence \( \zeta_N(s) = \prod_{p \nmid N} \left( 1 + c_{p} \frac{1}{p^s} + c_{2p} \frac{1}{2p^s} + \cdots \right) \)

for some $c_p, c_{2p} \in \mathbb{Z}^+$ and $\text{Re}(s) > 1$.

\[\Rightarrow \zeta_N(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \text{ for some } b_n \in \mathbb{Z}^+ \text{ and } \text{Re}(s) > 1.\]

**Summary of what we have got**: Some $b_n \in \mathbb{Z}^+$ such that

1. \( \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) converges absolutely for $\text{Re}(s) > 1$.
2. \( \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) diverges for $s = \frac{1}{\varphi(N)}$.
3. \( \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) has a holomorphic extension on $\text{Re}(s) > 0$.

The following theorem due to Landau gives us a contradiction.

**Theorem**: Suppose $a_n \in \mathbb{R}^+$. If \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) is convergent on $\text{Re}(s) > \rho$

and has a holomorphic extension on an open disk around $\rho$, then

\( \exists \varepsilon > 0 \) st. \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) converges on $\text{Re}(s) > \rho - \varepsilon$.

Before we prove the above theorem, let's see how it gives us a contradiction. Let \( s_0 := \inf \{ \frac{1}{\varphi(N)} \text{ s.t. } \sum_{n=1}^{\infty} \frac{b_n}{n^s} \text{ converges} \} \).

By (2), \( s_0 \geq \frac{1}{\varphi(N)} > 0 \).
By (2), \( s_2 \geq \frac{1}{\varphi(N)} > 0 \).

So \( \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) cannot have even a continuous extension to \( \text{Re}(s) > 0 \), let alone holomorphic.

Proof of Landau’s theorem. First observe that if for some \( x \in \mathbb{R} \)
\( \sum_{n=1}^{\infty} \frac{a_n}{n^x} < \infty \), then \( \sum_{n=1}^{\infty} \frac{a_n}{n^x} \) is absolutely convergent, and uniformly on compact sets on \( \text{Re}(s) > x \).

\( \text{Re}(s) > x \Rightarrow \left| \frac{a_n}{n^x} \right| \leq \frac{a_n}{n^x} \Rightarrow \text{By comparison,} \sum_{n=1}^{\infty} \frac{a_n}{n^x} \) is abs. conv.

Using the fact that \( \left| \frac{1}{n^s_1} - \frac{1}{n^s_2} \right| = \left| \int_{n_1}^{n_2} (-\ln n) n^{-\frac{s+s_2}{2}} \, dz \right| \leq \ln(n) \int_{n_1}^{n_2} n^{-\text{Re}(z)} \, |dz| \leq \ln(n) \cdot \min\{\text{Re}(s_1), \text{Re}(s_2)\} |n^s_1 - n^s_2|, \)

it is easy to see that, on \( \text{Re}(s) > x \), \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) converges uniformly on compact sets. And so on \( \text{Re}(s) > x \) it is holomorphic.

Let \( f \) be the holomorphic function on \( X \), and \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) for \( \text{Re}(s) > \rho \).

For simplicity, let’s shift the whole thing to assume \( \rho = 0 \), i.e. let \( g(s) = f(s - \rho) \).

So \( g \) is holomorphic on \( Y \) and

\[ g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{cohere} \quad c_n = \frac{a_n}{n^\rho} \geq 0. \]

For some \( \epsilon > 0 \), the open disk centered at 1
with radius $1+2\varepsilon$ is a subset of $Y$. So $g(s)$ is equal to its Taylor expansion on this open disk; and Taylor series converges abs.\\
$|s-1| < 1+2\varepsilon \implies g(s) = \sum_{k=0}^{\infty} \frac{g^{(k)}(1)}{k!} (s-1)^k$.

$\frac{d}{ds} (n^{-s}) = \frac{d}{ds} (e^{-\ln(n)s}) = -(\ln n) e^{-\ln(n)s} = -(\ln n) n^{-s}$.

$\implies \frac{d^k}{(ds)^k} (n^{-s}) = (-\ln n)^k n^{-s}$.

$\implies g^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(-\ln n)^k}{n} C_n n^{-s}$ on $\text{Re}(s) > \sigma$.

$\implies g^{(k)}(1) = \sum_{n=1}^{\infty} \frac{(-\ln n)^k}{n} C_n$.

$\implies g(s) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{n=1}^{\infty} \frac{(-\ln n)^k}{n} C_n \right) (s-1)^k$ if $|s-1| < 1+2\varepsilon$.

$\implies g(-\varepsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{n=1}^{\infty} \frac{(\ln n)^k}{(1+\varepsilon)^k} C_n \right)$

$= \sum_{n=1}^{\infty} \frac{1}{k!} \left( \sum_{k=0}^{\infty} \frac{(1+\varepsilon)^k}{(1+\varepsilon)^k} \ln n \right) \frac{C_n}{n}$

Since it is absolutely conv. $\sum_{n=1}^{\infty} \frac{C_n}{n^{-\varepsilon}}$.

So by the first part $\sum_{n=1}^{\infty} \frac{C_n}{n^\varepsilon}$ is absolutely convergent on $\text{Re}(s) > -\varepsilon \implies \sum_{n=1}^{\infty} \frac{a_n}{n^\varepsilon}$ is abs. conv. on $\text{Re}(s) > \rho - \varepsilon$. $\blacksquare$