You have already seen the main steps of Merten–Newman proof of Prime Number Theorem in your HW assignments. Since it is an important result, I would like to quickly go over its proof again.

. $\zeta(s)\neq 0$ if $\mathrm{Re}(s)\geq 0$

. If $\zeta(1+it_0)=0$, then $|\zeta(\sigma+it_0)|\ll|\sigma-1|$. (for $1<\sigma<2$)

[ζ is holomorphic at $1+it_0 \Rightarrow \lim_{\sigma\to 1}\frac{\zeta(\sigma+it_0)-\zeta(1+it_0)}{\sigma-1}$
$=\zeta'(1+it_0)$.]

. $\zeta(s)=\frac{1}{s-1}+\phi(s)$ where ϕ is holomorphic on $\mathrm{Re}(s)>0$.

$\Rightarrow |\zeta(\sigma)|\ll\frac{1}{|\sigma-1|}$ (for $1<\sigma<2$)

. $|\zeta(\sigma+2it_0)|\ll 1$ as ζ is continuous on a neighborhood of $1+2it_0$.

$$\Rightarrow |\zeta(\sigma)^3\,\zeta(\sigma+it_0)^4\,\zeta(\sigma+2it_0)|\xrightarrow[\sigma\to 1^+]{} 0. \quad \text{(I)}$$

. On the other hand,

$\ln|\zeta(s)|=\mathrm{Re}(\ln\zeta(s))$ [Notice that the real part of a complex logarithmic fuction of a holomorphic function does NOT depend on the choice of the logarithmic function.]

$$\mathrm{Re}(\ln\zeta(s))=\sum_p \mathrm{Re}\left(-\mathrm{Ln}\left(1-\frac{1}{p^s}\right)\right)$$
$$=\sum_p \mathrm{Re}\left(\sum_{m=1}^{\infty}\frac{1}{m p^{ms}}\right)=\sum_{n=1}^{\infty}\lambda(n)\,\mathrm{Re}(n^{-s})$$

where $\lambda(n)=\begin{cases}1/m & \text{if } n=p^m\end{cases}$

where $\lambda(n) = \begin{cases} 1/m & \text{if } n = p^m \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \ln |\zeta(\sigma)^3 \cdot \zeta(\sigma+it)^4 \cdot \zeta(\sigma+2it)| = 3\,\mathrm{Re}(\ln \zeta(\sigma)) + 4\,\mathrm{Re}(\ln \zeta(\sigma+it)) + \mathrm{Re}(\ln \zeta(\sigma+2it))$$

$$= \sum_{n=1}^{\infty} \lambda(n) \left[3\,\mathrm{Re}(n^{-\sigma}) + 4\,\mathrm{Re}(n^{-\sigma-it}) + \mathrm{Re}(n^{-\sigma-2it}) \right]$$

$\mathrm{Re}(n^{-\sigma-it}) = n^{-\sigma} \cos[(\ln n)t]$.

$$= \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}} \left[3 + 4\cos((\ln n)t) + \underbrace{\cos(2(\ln n)t)}_{2\cos^2((\ln n)t) - 1} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2\lambda(n)}{n^{\sigma}} (\cos[(\ln n)t] + 1)^2 .$$

In particular, $|\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \geq 1$ (II)

for any $\sigma > 1$ and $t \in \mathbb{R}$.

(II) clearly contradicts (I).

Logarithm of $\zeta(s)$.

By the above result, $\overbrace{(s-1)\zeta(s)}^{h(s)}$ is holomorphic and non-zero at any point of an open neighborhood U of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$. Hence

$\exists g: U \to \mathbb{C}$ holomorphic s.t. $e^{g(s)} = h(s)$

$$\Rightarrow g'(s) e^{g(s)} = h'(s) \Rightarrow g'(s) = \frac{h'(s)}{h(s)}$$

$h'(s) = (s-1)\zeta'(s) + \zeta(s)$ (for $s \neq 1$)

$$\Rightarrow \frac{h'(s)}{h(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \quad (\text{for } s \neq 1)$$

$$\Rightarrow -\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \underbrace{g'(s)}_{\text{holomorphic on } U}.$$

Summation form of $\frac{\zeta'(s)}{\zeta(s)}$.

We have seen that using Euler product we get:

$$e^{-\sum_p \mathrm{Ln}(1-1/p^s)} = \zeta(s) \quad \text{for } \mathrm{Re}(s) > 1$$

$$\Rightarrow \quad \frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{d}{ds} \mathrm{Ln}\left(1-\frac{1}{p^s}\right)$$

$$= -\sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{d}{ds}\left(e^{-m(\ln p)s}\right)$$

$$= \sum_p \sum_{m=1}^{\infty} \frac{\ln p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \\ 0 & \text{otherwise}. \end{cases}$

$$= \sum_p \frac{\ln p}{p^s} + r(s)$$ where r is holomorphic on $\mathrm{Re}(s) > \frac{1}{2} + \varepsilon_0$.

In particular, $r(s)$ can be assumed to be bounded on U (this is an assumption on U.)

So altogether we get:

$\exists\, \phi : U \to \mathbb{C}$ holomorphic s.t.

for $\mathrm{Re}(s) > 1$, $\quad \sum_p \frac{\ln p}{p^s} = \frac{1}{s-1} + \phi(s)$.

Let $\psi(t) := \sum_{p \le t} \ln p$. We have proved that $\prod_{p \le n} p < 4^n$.

$\Rightarrow \psi(t) \ll t$. (III)

Using integration-by-part we have:

$$\sum_{p \le x} (\ln p)\, p^{-s} = \psi(x) \cdot x^{-s} + s \int_2^x \psi(t) \cdot t^{-(s+1)}\, dt$$

(III) $\Rightarrow$ For $\mathrm{Re}(s) > 1$,

$$\sum_p \frac{\ln p}{p^s} = s \int_2^{\infty} \frac{\psi(t)}{t^{s+1}}\, dt.$$

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$$\Rightarrow \quad \Phi(s) = s\int_2 \frac{\psi(t)}{t^{s+1}}\,dt - \frac{1}{s-1}$$

$$\Rightarrow \quad \Phi(s) - 1 = s\int_2^\infty \frac{\psi(t)}{t^{s+1}}\,dt + \frac{s}{1-s}$$

$$\Rightarrow \quad \frac{\Phi(s)-1}{s} = \int_1^\infty \frac{\psi(t)}{t^{s+1}}\,dt - \int_1^\infty \frac{1}{t^s}\,dt$$

$$= \int_1^\infty \frac{\psi(t)-t}{t^{s+1}}\,dt.$$

So:

$\exists$ a holomorphic function $g_1 : U \to \mathbb{C}$ s.t.

$$g_1(s) = \int_1^\infty \frac{\psi(t)-t}{t^{s+1}}\,dt \quad \text{for } \mathrm{Re}(s) > 1.$$

The following analytic theorem is crucial:

Theorem. Suppose $f(t)$ is bounded and locally integrable.

Suppose $g(z) := \int_0^\infty f(t)\, e^{-tz}\,dt$ for $\mathrm{Re}(z) > 0$

has a holomorphic extension to $\mathrm{Re}(z) \geq 0$. Then

$$\int_0^\infty f(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty f(t)\,dt = g(0).$$

We have:

$$\int_1^\infty (\psi(t)-t)\, e^{-(\ln t)(s+1)}\,dt \quad \text{for } \mathrm{Re}(s) > 1$$

has holomorphic extension to $\mathrm{Re}(s) \geq 1$.

Let $t = e^x \Rightarrow dt = e^x\,dx \Rightarrow$

$$\int_0^\infty (\psi(e^x) - e^x)\cdot e^{-x(s+1)} \cdot e^x\,dx = \int_0^\infty (\psi(e^x) - e^x)\, e^{-xs}\,dx$$

Let $z = s-1$. So $\displaystyle\int_0^\infty \frac{\psi(e^x) - e^x}{e^x} \cdot e^{-xz}\,dx$

where $\frac{\psi(e^x) - e^x}{e^x}$ is bounded and locally integrable;

the integral is defined for $\mathrm{Re}(z) > 0$ and

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has holomorphic extension to Re(z) ≥ 0.

So by the Analytic Theorem,

$$\int_0^\infty \frac{\psi(e^x)-e^x}{e^x}\,dx < \infty$$

$$\left(t=e^x \Rightarrow dt = e^x\,dx \Rightarrow \frac{dt}{t}=dx.\right)$$

$$\Rightarrow \boxed{\int_1^\infty \frac{\psi(t)-t}{t^2}\,dt < \infty.}$$

Proof of PNT.

If $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} \neq 1$, then

$\exists\, \varepsilon_0 > 0$ and a sequence $t_1 < t_2 < \cdots$ s.t.

$$\left|\frac{\psi(t_n)}{t_n} - 1\right| > \varepsilon_0.$$

So passing to a subsequence, if needed, we can and will assume that either $\frac{\psi(t_n)}{t_n} > 1+\varepsilon_0 = \lambda > 1$ for any n

or $\frac{\psi(t_n)}{t_n} < 1-\varepsilon_0 = \lambda < 1$ for any n.

In the first case, we have

$$\int_{t_n}^{\lambda t_n} \frac{\psi(t)-t}{t^2}\,dt \geq \int_{t_n}^{\lambda t_n} \frac{\lambda t_n - t}{t^2}\,dt$$

$$= \lambda t_n\left(\frac{1}{t_n} - \frac{1}{\lambda t_n}\right) - \left[\ln(\lambda t_n) - \ln t_n\right]$$

$$= (\lambda-1) - \ln\lambda > 0. \quad \text{(IV)}$$

On the other hand, since $\int_1^\infty \frac{\psi(t)-t}{t^2}\,dt < \infty$, we have

$$\lim_{n\to\infty} \int_{t_n}^{\lambda t_n} \frac{\psi(t)-t}{t^2}\,dt = 0,$$ which contra. (IV)

In the second case, we have

$$\int_{\lambda t_n}^{t_n} \frac{2\psi(t)-t}{t^2}\,dt < \int_{\lambda t_n}^{t_n} \frac{\lambda t_n - t}{t^2}\,dt = -\left[(\lambda-1)-\ln\lambda\right] < 0 \quad \text{(V)}$$

Again $\lim_{n\to\infty} \int_{\lambda t_n}^{t_n} \frac{2\psi(t)-t}{t^2}\,dt = 0$, which contradicts (V).

Some of the consequences of the above results.

We have proved that there is an open neighborhood U of $\{z\in\mathbb{C} \mid \mathrm{Re}(z)\geq 0\}$ such that $g_1(s)=(s-1)\zeta(s)$ is holomorphic on U and $g_1(s)\neq 0, \forall s\in U$.

Hence $g_2(s)=\frac{1}{g_1(s)}$ is holomorphic on U, and so $g_3(s)=(s-1)g_2(s)$ is holomorphic on U and $g_3(1)=0$.

Notice that $g_3(s)=\frac{1}{\zeta(s)}$ for $s\in U\setminus\{1\}$.

You have proved in your HW assignments that if

f_1 and f_2 are multiplicative functions
$\sum_{n=1}^{\infty} \frac{f_i(n)}{n^s}$ are absolutely convergent for $\mathrm{Re}(s)>\rho_0$ $\Big\} \Rightarrow$

$$\sum_{n=1}^{\infty} \frac{f_1 * f_2(n)}{n^s} = \left(\sum_{n=1}^{\infty}\frac{f_1(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{f_2(n)}{n^s}\right)$$

for $\mathrm{Re}(s)>\rho_0$.

In particular, for $\mathrm{Re}(s)>1$,

$$\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right) = \sum_{n=1}^{\infty}\frac{(\mu * \mathbb{1})(n)}{n^s}$$

$$= \sum_{n=1}^{\infty}\frac{I(n)}{n^s} = 1.$$

$$\Rightarrow \boxed{\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad \text{for } \mathrm{Re}(s)>1.}$$

Let $M(x) := \sum_{n \le x} \mu(n)$. So

$$\sum_{n \le x} \frac{\mu(n)}{n^s} = \frac{M(x)}{x^s} + s\int_1^x \frac{M(t)}{t^{s+1}}\,dt$$

$\Rightarrow$ for $\mathrm{Re}(s) > 1$, $g_3(s) = \lim_{x \to \infty} \frac{M(x)}{x^s} + s\int_1^\infty \frac{M(t)}{t^{s+1}}\,dt$

$$\left|\frac{M(x)}{x^s}\right| \le \frac{x}{x^\sigma} = \frac{1}{x^{\sigma - 1}} \xrightarrow[x \to \infty]{} 0$$

$\Rightarrow$ for $\mathrm{Re}(s) > 1$, $g_3(s) = s\int_1^\infty \frac{M(t)}{t^{s+1}}\,dt$

$\Rightarrow$ for $\mathrm{Re}(s) > 1$, $\frac{g_3(s)}{s} = \int_0^\infty M(e^x)\, e^{-sx}\,dx$

$t = e^x \Rightarrow \frac{dt}{t} = dx$

$\Rightarrow$ $\frac{g_3(z+1)}{z+1} = \int_0^\infty \frac{M(e^x)}{e^x} \cdot e^{-zx}\,dx$ for $\mathrm{Re}(z) > 0$

has holomorphic extension to $\mathrm{Re}(z) \ge 0$

, and $\left|\frac{M(e^x)}{e^x}\right| \le 1$.

Hence by the Analytic Theorem

$$\int_0^\infty \frac{M(e^x)}{e^x}\,dx = \frac{g_3(1)}{1} = 0$$

So $\int_1^\infty \frac{M(t)}{t^2}\,dt = 0$. (VI)

<u>Theorem</u>. $\sum_{n \le x} \mu(n) = o(x)$, i.e. $\lim_{x \to \infty} \frac{M(x)}{x} = 0$.

<u>Proof</u>. If not, $\exists\, x_1 < x_2 < \cdots \to \infty$ and $1 > \varepsilon_0 > 0$ such that

$\left|\frac{M(x_i)}{x_i}\right| > \varepsilon_0$. Passing to a subsequence, if needed, we

have either $M(x_i) > \varepsilon_0 x_i$ for any i or

$M(x_i) < -\varepsilon_0 x_i$ for any i.

<u>Case 1.</u> $M(x_i) > \varepsilon_0 x_i \Rightarrow \forall\, x \le x_i$, $M(x) \ge M(x_i) - (x_i - x)$

$$\Rightarrow \int_{(1-\varepsilon_0)x_i}^{x_i} \frac{M(t)}{t^2}\,dt > \int_{(1-\varepsilon_0)x_i}^{x_i} \frac{t-(1-\varepsilon_0)x_i}{t^2}\,dt$$

$$= -\ln(1-\varepsilon_0) - (1-\varepsilon_0)x_i\left[\frac{1}{x_i} - \frac{1}{(1-\varepsilon_0)x_i}\right]$$

$$= -\ln(1-\varepsilon_0) - \left[(1-\varepsilon_0) - 1\right]$$

$$= \varepsilon_0 - \ln(1-\varepsilon_0) > 0,$$

which contradicts (IV) as before.

<u>Case 2.</u> $M(x_i) < -\varepsilon_0 x_i \Rightarrow \forall\ x \geq x_i,$

$$M(x) \leq M(x_i) + (x - x_i)$$

$$< x - (1+\varepsilon_0)x_i.$$

$$\Rightarrow \int_{x_i}^{(1+\varepsilon_0)x_i} \frac{M(t)}{t^2}\,dt < \int_{x_i}^{(1+\varepsilon_0)x_i} \frac{t-(1+\varepsilon_0)x_i}{t^2}\,dt$$

$$= \ln(1+\varepsilon_0) - (1+\varepsilon_0)x_i\left[\frac{1}{x_i} - \frac{1}{(1+\varepsilon_0)x_i}\right]$$

$$= \ln(1+\varepsilon_0) - \left[(1+\varepsilon_0) - 1\right]$$

$$= -\varepsilon_0 + \ln(1+\varepsilon_0) < 0$$

which contradicts (VI). ■

<u>Theorem.</u> $\sum_{n\leq x} \frac{\mu(n)}{n} = o(1)$, i.e. $\lim_{x\to\infty} \sum_{n\leq x} \frac{\mu(n)}{n} = 0$.

<u>Proof</u>. By integration-by-part

$$\sum_{n\leq x} \frac{\mu(n)}{n} = M(x)/x + \int_1^x \frac{M(t)}{t^2}\,dt$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \lim_{x\to\infty} \frac{M(x)}{x} + \int_1^{\infty} \frac{M(t)}{t^2}\,dt$$

$$= 0,$$ by the previous Theorem and (VI).

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Möbius Randomness Law: μ does NOT correlate with any function of "low complexity", i.e.

$$\sum_{n\le x} \mu(n) f(n) = o\Big(\sum_{n\le x} |f(n)|\Big).$$

[Warning: this is not a theorem, this is just a type of guideline.]

Remark. $M(x) = O(x^{1/2+\varepsilon})$ is equivalent to Riemann Hypothesis.