1. Let $\psi(x) = \sum_{\gamma \leq x} \ln \gamma$.

(a) Prove that $\psi(x) = O(x)$.

(b) Prove that $\psi(x) \sim x$ $\iff$ Prime Number Theorem, i.e.

$$\pi(x) \sim \frac{x}{\ln x}.$$ 

[Hint: (a) We have already proved a stronger result.]

(b) We have proved the following:

- $\{\lambda_n\}$ strictly increasing seq of positive integers
- $\{c_n\}$ a sequence of complex numbers
- $f: \mathbb{R}^+ \to \mathbb{C}$ differentiable

Let $C(t) := \sum_{\lambda_n \leq t} c_n$. Then

$$\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x) f(x) - \int_1^x C(t) f'(t) \, dt.$$ 

- Let $\lambda_n = p_n$ be the $n$th prime number, $f(x) = \frac{1}{\ln x}$, and
  $$c_n = \frac{\ln p_n}{\ln x}.$$ 

- You have proved it before that $\int_1^x \frac{1}{(\ln t)^2} \, dt \ll \frac{x}{(\ln x)^2}$.]

2. Prove that there is a holomorphic function $h$ on an open neighborhood $U$ of $\mathbb{C}$ such that $\Re e(z) \geq 1$ such that

$$h(s) + \frac{1}{s-1} = \sum_{p} \frac{\ln p}{p^s}$$

for $\Re e(s) > 1$.

[Hint: In the previous HW, you proved that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Delta(n)}{n^s}$$

for $\Re e(s) > 1$

$$= \sum_{p} \frac{\ln p}{p^s} + \sum_{p^m \geq 2} \frac{\ln(p)}{p^{ms}}$$

for $\Re e(s) > 1$.]

1. Show that \( g_1(s) := \sum_{p, m \geq 2} \frac{\ln p}{p^m} \) is holomorphic on an open neighborhood of \( \{ z \in \mathbb{C} \mid \text{Re}(z) > \frac{1}{8} \} \).

2. \( \sum_{p} \frac{\ln p}{p^s} \) is holomorphic on \( \text{Re}(s) > 1 \).

3. \( \exists \) a holomorphic function \( g_2 \) on an open neighborhood \( U \) of \( \{ z \in \mathbb{C} \mid \text{Re}(z) > \frac{1}{8} \} \) such that \( \frac{\zeta(s)}{\zeta(0)} = \frac{1}{s-1} + g_2(s) \)

   [This is the important step. You should use:
   a. \( \zeta(s) = \frac{1}{s-1} + \phi(s) \) where \( \phi \) is holomorphic on \( \text{Re}(s) > 0 \).
   b. \( \zeta(s) \neq 0 \) if \( \text{Re}(s) > 1 \) (Previous HW."
   )]

4. Let \( h(s) := g_2(s) - g_1(s) \) on \( U \), and conclude \( \sum_{p} \frac{\ln p}{p^s} = \frac{1}{s-1} + h(s) \) for \( \text{Re}(s) > 1 \).

3\( \sum_{p} \frac{\ln p}{p^s} = \zeta(s) \sum_{t=1}^{\infty} \frac{\varphi(t)}{t^{s+1}} \) for \( \text{Re}(s) > 1 \).

   [Hint: Use the mentioned "integration by parts" in the hint of problem 1. And \( \varphi(t) = 0 \) if \( 1 \leq t < 2 \).

6. Conclude \( \int_{1}^{\infty} \frac{\varphi(t)-t}{t^{s+1}} \, dt = \frac{h(s)-1}{s} \) for \( \text{Re}(s) > 1 \),

   and so the LHS has a holomorphic extension on \( U \),

   a neighborhood of \( \{ z \in \mathbb{C} \mid \text{Re}(z) > \frac{1}{8} \} \).

Suppose as in Landau theorem we manage to use the holomorphic extension of \( \int_{1}^{\infty} \frac{\varphi(t)-t}{t^{s+1}} \, dt \) to conclude this is a convergent integral even at \( s=1 \). I.e.

\[
\int_{1}^{\infty} \frac{\varphi(t)-t}{t^{2}} \, dt < \infty.
\]

4. Prove \( \varphi(x) \sim x \).
Hint. We have to show
\[ \forall \varepsilon > 0, \quad x \geq 1 \Rightarrow \frac{1}{x} \leq \psi(x) \leq (1+\varepsilon)x \]
If not, then either there is a sequence
\[ x_1 < x_2 < \cdots, \quad x_n \to \infty \quad \text{and} \quad \psi(x_n) > (1+\varepsilon)x_n \]
or there is a sequence \( y_1 < y_2 < \cdots, \quad y_n \to \infty \quad \text{and} \quad \psi(y_n) < (1-\varepsilon)y_n \).

Now we would like to get a contradiction:
\[
\int_{x_n}^{(1+\varepsilon)x_n} \frac{\psi(t)-t}{t^2} \, dt \geq \int_{x_n}^{(1+\varepsilon)x_n} \frac{(1+\varepsilon)x_n-t}{t^2} \, dt
\]
\[ = \varepsilon - \ln(1+\varepsilon) \quad \square \]
Since \( \int_1^\infty \frac{\psi(t)-t}{t^2} \, dt \) is convergent,
\[
\lim_{n \to \infty} \int_{x_n}^{(1+\varepsilon)x_n} \frac{\psi(t)-t}{t^2} \, dt = \square, \quad \text{which contradicts} \quad \square \quad \square \]

The other part is similar.

So modulo \( \S \) you have proved the **Prime Number Theorem**. Statements similar to \( \S \) is called Tauberian Theorems.

Here we used Newman’s simple proof of prime number theorem. I have used the following article by Zagier:

Zagier, Newman’s Short Proof of Prime Number Theorem,