Wednesday, May 25, 2016 8:46 PM

1. Prove that $\sum_{n\le x} \mu(n)\, n = o(x^2)$, i.e. $\lim_{x\to\infty} \dfrac{\sum_{n\le x}\mu(n)\,n}{x^2} = 0$.

[Hint. Let $M(x) = \sum_{n\le x} \mu(n)$. You are allowed to use $M(x) = o(x)$, i.e. $\lim_{x\to\infty} \dfrac{M(x)}{x} = 0$.

Use (discrete) integration-by-part.]

2. Suppose G is a finite abelian group. Suppose

$$0 \longrightarrow G_1 \xrightarrow{\ \phi\ } G \xrightarrow{\ \pi\ } G_2 \longrightarrow 0$$

is a short exact sequence, i.e.

π is onto, ϕ is injective, and $\mathrm{im}(\phi) = \ker(\pi)$.

Prove that the following are equivalent:

ⓐ $\exists\, s: G_2 \longrightarrow G$, a group homomorphism, such that

$$\pi \circ s = \mathrm{id}_{G_2}.$$

ⓑ $\exists\, f: G_1 \oplus G_2 \longrightarrow G$, a group isomorphism, such that $f(g_1, 0) = \phi(g_1)$ and $\pi(f(g_1, g_2)) = g_2$.

ⓒ $\exists\, \psi: G \longrightarrow G_1$, a group homomorphism, such that

$$\psi \circ \phi = \mathrm{id}_{G_1}.$$

[Hint. • Suppose ⓐ holds; let $f: G_1 \oplus G_2 \longrightarrow G$,

$$f(g_1, g_2) = \phi(g_1) + s(g_2).$$

Prove f is an isomorphism. [ⓐ $\Rightarrow$ ⓑ]

- Let $\psi = pr_1 \circ f^{-1} \circ \phi$, where pr_1 is the proj. to the first component. [ⓑ $\Rightarrow$ ⓒ].
- Let $s(g_2) := f(0, g)$. [ⓑ $\Rightarrow$ ⓐ].
- Let $h: G \longrightarrow G_1 \oplus G_2$, $h(g) = (\psi(g), \pi(g))$, prove that h is a group isomorphism.]

3 ⓐ Prove that $\widehat{G_1 \oplus G_2} \simeq \widehat{G_1} \oplus \widehat{G_2}$ where G_1 and G_2 are two finite abelian groups.

ⓑ Conclude that, if $G \simeq \bigoplus_{i=1}^{k} \mathbb{Z}/_{n_i \mathbb{Z}}$, then $G \simeq \widehat{G}$.

4. Suppose G is a finite abelian group. In the lecture, we used the linear transformation

$$\rho_g : L^2(G) \to L^2(G), \quad \rho_g(f)(g') := f(g'g).$$

to show $\prod_{\chi \in \widehat{G}} (T - \chi(g)) = (T^{o(g)} - 1)^{|G|/o(g)}$.

Use the same idea and use trace to conclude:

$$\sum_{\chi \in \widehat{G}} \chi(g) = 0 \quad \text{if } g \neq \text{identity of } G.$$

5. Let d be a square-free integer and $d \overset{4}{\equiv} 1$, $d > 1$.

Let $\chi_d : \mathbb{Z}^+ \to \{-1, 1, 0\}$ be

$$\chi_d(n) = \begin{cases} \left(\frac{d}{n}\right) & \text{if } \gcd(d,n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

ⓐ Show that χ_d is a Dirichlet character. (Hint: use quadratic reciprocity.) Conclude that $L(\chi_d, s) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$ is convergent and holomorphic for $\mathrm{Re}(s) > 0$.

ⓑ Let $R(n) := \sum_{m|n} \chi_d(m)$. Prove that $R(n)$ is multiplicative.

ⓒ Prove that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \le x \\ \gcd(n,d)=1}} R(n) = \frac{\varphi(d)}{d} L(\chi_d, 1)$$

where $\varphi(d)$ is the Euler phi-function.

[Hint. $\sum_{n \le x} R(n) = \sum_{n < x} \sum_{m|n} \chi_d(m)$

$$\text{[Hint.} \sum_{\substack{n \le x \\ \gcd(n,d)=1}} R(n) = \sum_{\substack{n \le x \\ \gcd(n,d)=1}} \sum_{m|n} \chi_d(m)$$

$$= \sum_{\substack{m_1 m_2 \le x \\ \gcd(m_1 m_2, d)=1}} \chi_d(m_1)$$

Two parts : $m_1 \le \sqrt{x}$ or $m_2 < \sqrt{x}$

Part I $\displaystyle\sum_{\substack{m_1 \le \sqrt{x} \\ \gcd(m_1, d)=1}} \chi_d(m_1) \sum_{\substack{m_2 \le x/m_1 \\ \gcd(m_2, d)=1}} 1$

Part II $\displaystyle\sum_{\substack{m_2 < \sqrt{x} \\ \gcd(m_2, d)=1}} \sum_{\substack{\sqrt{x} < m_1 \le x/m_2 \\ \gcd(m_1, d)=1}} \chi_d(m_1)$

- Partial sum of non-trivial Dirichlet characters is $O(d)$.

$\Rightarrow$ Part II is $O(d\sqrt{x})$

- $\displaystyle\sum_{\substack{m \le y \\ \gcd(m,d)=1}} 1 = \frac{\varphi(d)}{d} y + O(d) \Rightarrow$

Part I is $\displaystyle\frac{\varphi(d)}{d} x \sum_{\substack{m_1 \le \sqrt{x} \\ \gcd(m_1, d)=1}} \frac{\chi_d(m_1)}{m_1} + O(d\sqrt{x})$.]

6. Let p_n be the n^{th} prime number. Prove that

$$\lim_{n \to \infty} \frac{p_n}{n \ln n} = 1.$$

[Hint. By PNT, $\displaystyle\lim_{n \to \infty} \frac{\pi(p_n)}{p_n / \ln(p_n)} = 1$

$$\Rightarrow \lim_{n \to \infty} \frac{n}{p_n / \ln(p_n)} = 1. \qquad \circledast$$

Suppose for some $n_1 < n_2 < \dots$

$$p_{n_i} \ge (1+\varepsilon) n_i \ln n_i$$

$$\Rightarrow \frac{n_i}{p_{n_i} / \ln p_{n_i}} \le \frac{n_i \ln((1+\varepsilon) n_i \ln n_i)}{(1+\varepsilon) n_i \ln((1+\varepsilon) n_i)}$$

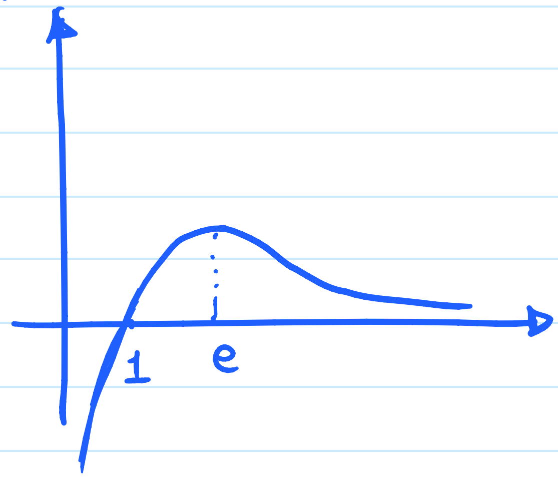
$$p_{n_i}/\ln p_{n_i} \quad - \quad (1+\varepsilon)n_i \cdot \ln((1+\varepsilon)n_i)$$

$$\xrightarrow[i \to \infty]{} \frac{1}{1+\varepsilon} \quad \text{which contradicts } \circledast$$

The other case is similar. (You are allowed to use the following graph without proof:

$$y = \frac{\ln x}{x}$$

In particular, it is decreasing for $x \geq e$.)



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