In the previous lecture we defined **conditional propositions** a.k.a. **implications**.

If \( P \), then \( Q \). \( P \implies Q \).

And its truth table is

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \implies Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

Since in mathematics we often deal with this type of propositions, let’s try to find new forms of such propositional form.

What does it mean for \( P \implies Q \) to fail? For you to show me this implication fails, you have to provide a situation where \( P \) is true and \( Q \) is false, which means

\[ \neg(P \implies Q) \equiv P \land (\neg Q) \]

Let’s double check this using the truth table.
<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>\neg Q</th>
<th>P \Rightarrow Q</th>
<th>\neg (P \Rightarrow Q)</th>
<th>P \land (\neg Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

So by de Morgan’s law we have
\[ P \Rightarrow Q \equiv \neg (P \land \neg Q) \equiv \neg P \lor Q \]
\[ \equiv Q \lor (\neg P) \]
\[ \equiv (\neg Q) \Rightarrow (\neg P) . \]

- \( \neg Q \Rightarrow \neg P \) is called the **contrapositive** of \( P \Rightarrow Q \), and it is a useful method to prove things.
- Before we see some examples, let me warn you that \( P \Rightarrow Q \not\equiv Q \Rightarrow P \) is called the **converse** of \( P \Rightarrow Q \).

**Ex. (Kenken)**

```
  3+2 4+1 1+3
  1 3 2
  2 3 1
```

- In the blue box we can have either \[
\begin{array}{c}
1 \\
3
\end{array}
\] or \[
\begin{array}{c}
3 \\
1
\end{array}
\].

If the second case happens, we would have two 3’s in a row, which is a contradiction. So the first case happens.
a row, which is a contradiction. So the first case happens.
In the yellow box there are two possible cases \( \frac{1}{2} \) or \( \frac{2}{1} \).
If the first case happens, we would get two 1s in a row, which is a contradiction. Hence the second case is true.

The only remaining possibility for \( \frac{1}{2} \) is 3.
Similarly we have that \( \frac{2}{1} \) is 2.
Using the same logic we have that \( \frac{1}{2} \) and \( \frac{2}{1} \) are 2 and 1, respectively.

In this game, you see how we use case-by-case proof together with proof by contradiction together in our daily games or decisions.

Def. Suppose \( m \) and \( n \) are two integers. We say \( m \) divides \( n \) if for some integer \( k \) we have
\[
    n = mk.
\]
(we also say \( m \) is a divisor of \( n \), or \( n \) is a multiple of \( m \)) we denote it by \( m \mid n \).
n is a multiple of m) We denote it by \( m \mid n \).

**Ex.** \( 1 \mid n \) for any integer \( n \).

**Pf.** For any integer \( n \), \( n = (n)(1) \). So \( n \) is a multiple of 1. ■

**Ex.** For non-zero integers \( a \) and \( b \), \( a \mid b \implies |a| \leq |b| \).

**Pf.** \( a \mid b \implies \) for some integer \( k \), \( b = ak \)

\[ \implies |b| = |a||k|. \]

**Claim** \( k \neq 0 \).

**Pf.** of claim. Suppose to the contrary that \( k = 0 \). Then

\[ b = (a)(0) = 0, \] which contradicts the assumption that \( b \) is non-zero.

Since \( k \) is a non-zero integer, we have \( |k| \geq 1 \).

Hence \( |b| = |a||k| \geq |a| \) as \( |b| \geq 0 \). ■

**Warning.** By *multiple*, we mean integer multiple. We are NOT allowed to multiply by fractions.

We also discussed that, if \( P \implies Q \) and \( Q \implies P \) are true, then \( P \) and \( Q \) are equivalent. And we showed this using the truth table:

\[
\begin{array}{c|c|c|c|c|}
P & Q & P \implies Q & Q \implies P & (P \implies Q) \land (Q \implies P) \\
T & T & T & T & T \\
T & F & F & T & T \\
F & T & T & F & T \\
F & F & T & T & F \\
\end{array}
\]

both are true.
<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

both are true.

both are false.