In the previous lecture we were proving:

**Theorem.** For any integer \( n \), \( n \) is odd \( \iff \) \( n = 2k+1 \) for some integer.

We proved \( (\Rightarrow) \). For \( (\Leftarrow) \) look at my previous note.

**Corollary.** For any integers \( m \) and \( n \), \( mn \) is odd if and only if \( m \) and \( n \) are odd.

\[ \text{Pf.} \ (\Rightarrow) \]

<table>
<thead>
<tr>
<th>Given</th>
<th>Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( mn ) is odd</td>
<td>( m ) is odd \ and ( n ) is odd</td>
</tr>
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**Proof by contradiction**

<table>
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<tr>
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<tr>
<td>( 2 \mid mn )</td>
<td>Contradiction</td>
</tr>
<tr>
<td>( \neg(2 \mid m \land 2 \mid n) )</td>
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We need to show **two** things
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\[
\begin{array}{|c|c|}
\hline
\text{Given} & \text{Goal} \\
\hline
2 \mid mn & \text{Contradiction} \\
2 \mid m & \\
\hline
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|}
\hline
\text{Given} & \text{Goal} \\
\hline
2 \mid mn & \text{Contradiction} \\
2 \mid n & \\
\hline
\end{array}
\]

They look the same: changing the roles of \( m \) and \( n \) we get the same statement.

In this kind of scenarios, we say “by symmetry it is enough to show that…” or “without loss of generality we can and will assume that…”

\[
2 \mid m \implies m = 2k \text{ for some integer } k
\]

\[
\implies mn = 2kn \text{ (and } kn \text{ is an integer)}
\]

\[
\implies 2 \mid mn
\]

which contradicts our assumption that \( mn \) is odd.

\[
(\Leftarrow)
\begin{array}{|c|c|}
\hline
\text{Given} & \text{Goal} \\
\hline
2 \mid m \land 2 \mid n & mn \text{ is odd} \\
\hline
\end{array}
\]

\[
2 \mid m \implies m = 2k+1 \text{ for some integer } k
\]

\[
2 \mid n \implies n = 2k'+1 \text{ for some integer } k'
\]

So we have

\[
mn = (2k+1)(2k'+1) = 4kk'+2k+2k'+1
\]
\[ = 2 \left( \sqrt{2k'k + k + k'} \right) + 1 \]

is an integer

\[ \text{Hence } mn = 2k'' + 1 \text{ for some integer } k''. \]

Therefore \( mn \) is odd. \( \blacksquare \)

Ex. Does the equation \( 14m - 49n = 1 \) have an integer solution?

Solution. No, it does NOT.

Suppose to the contrary that it does have a solution.

So \( 14m - 49n = 1 \) for some integers \( m \) and \( n \).

Therefore \( 7(2m - 7n) = 1 \), which implies

\[ 7 \mid 1 \]

as \( 2m - 7n \) is an integer. By our earlier result we should have \( |7| \leq |1| \) which is a contradiction. \( \blacksquare \)

Remark. Equation \( ax + by = 1 \) has an integer solution if and only if \( a \) and \( b \) have no common divisors except \( \pm 1 \).

We will prove this later, but \( \Rightarrow \) is easy and the same argument as above works.
Lemma. Let $a, b, c$ be integers. If for some integers $x$ and $y$, $ax + by = c$, and $d$ is a common divisor of $a$ and $b$, then $d \mid c$.

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<td>$ax + by = c$</td>
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<tr>
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<td></td>
</tr>
<tr>
<td>$d \mid b$</td>
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$d \mid a \implies$ for some integer $k$, $a = dk$ \(\text{I}\)

$d \mid b \implies$ for some integer $k'$, $b = dk'$ \(\text{II}\)

By \(\text{I}\), \(\text{II}\), and our assumption,

$$c = ax + by = dkx + dk'y$$

$$\implies c = d(kx + ky)$$

is an integer

$$\implies d \mid c$$.

Ex. There are no integers $m$ and $n$ such that

$$66m - 110n = 2$$.

Pf. Suppose to the contrary that there are such integers.

Then $11(6m - 10n) = 2$, which implies $11 \mid 2$.

Therefore we should have $11 \leq 2$, which is a contradiction.
An alternative approach: Suppose to the contrary that there are such integers m and n. So, by the above lemma, since 11 is a common divisor of 66 and 110, it should be a divisor of 2. Hence we should have $11 \leq 2$, which is a contradiction. ■